

# Quantum systems with identical energy spectra

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An algorithm establishing an analytic relation between bound states of different quantum systems is proposed for two and three dimensions. An entire class of new, exactly solvable multidimensional problems can be constructed by using this algorithm.

1. In one-dimensional quantum mechanics, there is a beautiful recipe, called the “method of factorization,” for forming a correspondence between any potential with known bound states and a series of new potentials, for which the energies and explicit expressions for the wave functions is also known.<sup>1</sup> This method<sup>1)</sup> was used, for example, to construct exactly solvable potentials,<sup>3</sup> and recently, also for constructing solutions of nonlinear equations.<sup>4</sup> In this paper we propose a generalization of the method of factorization to the case of a large number of dimensions.

2. We recall briefly the main points of the method of factorization, which are necessary to generalize it to higher dimensions. Let us assume that for a one-dimensional ( $d = 1$ ) potential  $V^{(0)}(x)$  the wave functions of the bound states  $\Psi_N^{(0)}$  with energies  $E_N^{(0)}$  ( $N = 0, 1, 2, \dots$ ) are known:

$$\begin{aligned} H^{(0)} \Psi_N^{(0)} &\equiv \left[ -\frac{1}{2} \partial^2 + V^{(0)}(x) \right] \Psi_N^{(0)} \\ &= E_N^{(0)} \Psi_N^{(0)}; \quad E_0^{(0)} \equiv 0. \end{aligned}$$

$H^{(0)}$  can always be represented in a factorized form

$$H^{(0)} = Q^+ Q^-, \quad (1)$$

$$Q^\pm \equiv \frac{1}{\sqrt{2}} (\mp \partial + \partial \chi) = \mp \frac{1}{\sqrt{2}} e^{\pm \chi} \partial e^{\mp \chi}; \quad \chi \equiv -\ln \Psi_0^{(0)}(x).$$

If a new Hamiltonian  $H^{(1)}$  is constructed according to the formula

$$H^{(1)} = Q^- Q^+ = H^{(0)} + [Q^-, Q^+] = -\frac{1}{2} \partial^2 + V^{(1)}(x);$$

$$V^{(1)} \equiv V^{(0)} + \partial^2 \chi, \quad (2)$$

then the following relations ("intertwining relations") are satisfied:

$$H^{(1)} Q^- = Q^- H^{(0)}, \quad Q^+ H^{(1)} = H^{(0)} Q^+. \quad (3)$$

By virtue of these relations **all the wave functions**  $\Psi_N^{(1)}$  of the Hamiltonian  $H^{(1)}$  are related to the wave functions  $\Psi_N^{(0)}$  (with the same energies) of the starting system

$$\Psi_N^{(1)}(x) = \frac{1}{\sqrt{E_{N+1}^{(0)}}} Q^- \Psi_{N+1}^{(0)}(x), \quad \Psi_N^{(0)}(x) = \frac{1}{\sqrt{E_N^{(0)}}} Q^+ \Psi_{N-1}^{(1)}(x)$$

$$(N = 1, 2, \dots).$$

The energy spectrum of  $H^{(1)}$  is obtained from the spectrum  $H^{(0)}$  by simply "crossing out" the lowest level  $E_0^{(0)}: E_N^{(1)} = E_{N+1}^{(0)} (N = 0, 1, 2, \dots); Q^- \Psi_0^{(0)} = 0$ .

This procedure for constructing new, exactly solvable potentials can be repeated, starting now from  $H^{(1)}$  and  $\Psi_0^{(1)}$ , etc. In so doing, a series of potentials  $V^{(n)}(x)$ , is obtained. From them the energy spectrum and the wave functions of bound states are found explicitly from the starting spectrum and wave functions. Thus, starting with the simplest potential  $V^{(0)}(x)$ —a rectangular well with infinite walls at  $x = \pm \pi/2$ —we immediately obtain the spectrum of bound states in potentials of the form  $V^{(n)}(x) = [n(n+1)]/[2\cos^2 x]$ .

3. To generalize the described algorithm to the case of two dimensions, it is necessary to construct, starting from the Hamiltonian  $H^{(0)} = -\frac{1}{2} \Delta^{(2)} + V^{(0)}(\mathbf{x}), \mathbf{x} = (x_1, x_2)$ , a Hamiltonian  $H^{(1)}$ , which intertwines with  $H^{(0)}$  according to formulas analogous to (3), thereby assuring that the spectra and wave functions of  $H^{(0)}$  and  $H^{(1)}$  are related. We introduce the operators

$$Q_l^\pm = \frac{1}{\sqrt{2}} (\mp \partial_l + \partial_l \chi) = \mp \frac{1}{\sqrt{2}} e^{\pm \chi} \partial_l e^{\mp \chi}; \quad \partial_l \equiv \frac{\partial}{\partial x_l}; \quad l = 1, 2,$$

where  $\chi(\mathbf{x}) = -\ln \Psi_0^{(0)}$ . Thus  $H^{(0)}$  assumes the "factorized" form

$$H^{(0)} = Q_1^+ Q_1^- + Q_2^+ Q_2^- \equiv Q_l^+ Q_l^- \quad (E_0^{(0)} \equiv 0). \quad (4)$$

The Schrödinger operator  $H_{ik}^{(1)}$  with a  $2 \times 2$  matrix potential  $V_{ik}^{(1)}$

$$H_{ik}^{(1)} = \delta_{ik} H^{(0)} + [Q_i^-, Q_k^+] = -\frac{1}{2} \Delta^{(2)} \delta_{ik} + V_{ik}^{(1)};$$

$$V_{ik}^{(1)}(\mathbf{x}) = \delta_{ik} V^{(0)} + \partial_i \partial_k \chi \quad (5)$$

by virtue of the commutation relations  $[Q_i^-, Q_k^+] = \partial_i \partial_k \chi$ ,  $[Q_i^+, Q_k^+] = 0$  intertwines with  $H^{(0)}$  (repeated indices  $i, k = 1, 2$ , are summed):

$$H_{ik}^{(1)} Q_k^- = Q_i^- H^{(0)}, \quad Q_i^+ H_{ik}^{(1)} = H^{(0)} Q_k^+ \quad (6)$$

It follows that the operators  $Q_i^-$  transfer all eigenfunctions  $\Psi^{(0)}$  (except  $\Psi_0^{(0)}$ ) into two-component wave functions  $\Psi_k^{(1)}$ ,  $k = 1, 2$ , of the Hamiltonian  $H_{ik}^{(1)}$  with the same values of the energy

$$\Psi_k^{(1)}(\mathbf{x}) = \frac{1}{\sqrt{E^{(0)}}} Q_k^- \Psi^{(0)}.$$

However, in contrast to the one-dimensional case, in the potential  $V_{ik}^{(1)}$ , there is generally another level, whose wave functions satisfy the equation  $Q_k^+ \Psi_k^{(1)} = 0$ . Such wave functions can be represented in the form  $\Psi_k^{(1)} = \epsilon_{kl} Q_l^+ \tilde{\Psi}^{(0)}$ , where  $\tilde{\Psi}^{(0)}$  are the characteristic functions of another scalar Hamiltonian,

$$\tilde{H}^{(0)} = Q_i^- Q_i^+ = H^{(0)} + \Delta^{(2)} \chi. \quad (7)$$

It intertwines with the Hamiltonian  $H_{ik}^{(1)}$  via the operators  $P_i^\pm = \epsilon_{ik} Q_k^\mp$  analogously to (6), and its spectrum fits entirely into the spectrum of  $H^{(1)}$  (including the ground state of  $H^{(0)}$ , which always lies above  $E_0^{(0)} \equiv 0$ ). All bound states in the potential  $V_{ik}^{(1)}$  are obtained in the manner described above from the bound states of either the potential  $\tilde{V}^{(0)}$  or the potential  $\tilde{V}^{(0)} = V^{(0)} + \Delta^{(2)} \chi$ . The level with zero energy in the potential  $V_{ik}^{(1)}$ , as in the one-dimensional case, is missing. These properties of  $H_{ik}^{(1)}$  follow from its expansion as a direct sum of operators

$$H_{ik}^{(1)} = Q_i^- Q_k^+ + P_i^- P_k^+ \equiv \underline{H}_{ik}^{(1)} + \underline{H}_{ik}^{(1)}; \quad \underline{H}_{ik}^{(1)} \cdot \underline{H}_{kl}^{(1)} = 0.$$

The operators  $\underline{H}_{ik}^{(1)}$  and  $\underline{H}_{ik}^{(1)}$  do not have the correct kinetic term, but their spectra (except for  $E_0^{(0)} = 0$ ) coincide exactly with the spectra of  $H^{(0)}$  and  $\tilde{H}^{(0)}$ , respectively.

This result can also be formulated in a different manner: the energies of the bound states (except for  $E_0^{(0)} = 0$ ) in the two matrix potentials

$$\frac{1}{2} \begin{pmatrix} (\partial_i \chi)^2 - \partial_i^2 \chi & 0 \\ 0 & (\partial_i \chi)^2 + \partial_i^2 \chi \end{pmatrix}$$

$$\text{and } \frac{1}{2} \begin{pmatrix} (\partial_i \chi)^2 + \partial_1^2 \chi - \partial_2^2 \chi & 2\partial_1 \partial_2 \chi \\ 2\partial_1 \partial_2 \chi & (\partial_i \chi)^2 - \partial_1^2 \chi + \partial_2^2 \chi \end{pmatrix},$$

coincide, while the corresponding wave functions are related to one another via  $Q_i^\pm, P_i^\pm$ . The existence of such quantum systems with identical energy spectrum is a result of **hidden supersymmetry**. The Hamiltonian  $\tilde{H}^{(0)}, H^{(1)}, H^{(0)}$  are components of the supersymmetrical Hamiltonian  $H$  in subspaces with fixed number of fermions  $n = 0, 1, 2$

$$\hat{H} = \frac{1}{2} p_k^2 + \frac{1}{2} (\partial_k \chi)^2 + \frac{1}{2} (\partial_k^2 \chi) - (\partial_k \partial_l \chi) b_k^+ b_l^- = \{Q^+, Q^-\};$$

$$\{b_k^-, b_l^+\} = \delta_{kl}, \quad \{b_k^+, b_l^+\} = 0,$$

where the supercharge operators  $Q^\pm = Q_i^\pm b_i^\pm$  change the number of fermions by 1. The intertwining relations between  $H^{(0)}, \tilde{H}^{(0)}, H^{(1)}$  follow from  $[\hat{H}, Q^\pm] = 0$  [see (6)]. As in the one-dimensional case, the construction of Hamiltonians with equivalent spectra can be continued: starting with the potential  $V_{ik}^{(1)}$ , it is possible to construct a new  $4 \times 4$  matrix potential  $V^{(2)}$ , whose bound states are obtained from the states of the starting  $V_{ik}^{(1)}$  and additional  $V_{ik}^{(1)}$  potentials, etc. This algorithm and its relation to supersymmetrical quantum mechanics will be examined in greater detail in a separate paper.

4. The recipe for constructing  $H^{(1)}$  can be extended to higher dimensions  $d = 3, 4, \dots$ . Thus, for a three-dimensional system, the solutions of the equations  $Q_k^+ \Psi_k^{(1)} = 0$  have the form  $\Psi_k^{(1)} = \epsilon_{klm} Q_m^+ \Psi_l = P_{kl}^- \Psi_l$ , where  $\Psi_l(\mathbf{x})$  are the eigenfunctions (three-dimensional columns) of another matrix Hamiltonian  $\tilde{H}_{ik}^{(1)}$ . Both Hamiltonians  $H_{ik}^{(1)}$  and  $\tilde{H}_{ik}^{(1)}$  are separated into a direct sum of mutually orthogonal terms:

$$H_{ik}^{(1)} = Q_i^- Q_k^+ + P_{il}^- P_{kl}^+ \equiv \underline{H}_{ik}^{(1)} + \underline{\underline{H}}_{ik}^{(1)},$$

$$\tilde{H}_{ik}^{(1)} = Q_i^+ Q_k^- + P_{il}^+ P_{kl}^- \equiv \tilde{\underline{H}}_{ik}^{(1)} + \tilde{\underline{\underline{H}}}_{ik}^{(1)}.$$

The operators  $Q_i^\pm$  relate the eigenfunctions (with identical energy) of  $H^{(0)}$  and  $H^{(1)}$ , as well as of  $\tilde{H}^{(0)}$  and  $\tilde{H}^{(1)}$ , while the operators  $P_{il}^\pm$  relate the eigenfunctions of  $\underline{H}^{(1)}$  and  $H^{(1)}$ .

We shall examine the three-dimensional attractive Coulomb potential  $V^{(0)} = -\alpha/r$ , for which the spectrum  $E_N^{(0)} = -\alpha^2/2(N+1)^2$ ,  $N = 0, 1$ , and wave functions of bound states are known. It turns out that it is possible to determine exactly the spectrum  $E_N^{(1)} = -\alpha^2/2(N+2)^2$  and the wave functions for the matrix potential  $V_{ik}^{(1)} = -\alpha(x_i x_k)/r^3$ . Since the potentials  $\tilde{V}^{(0)} = \alpha/r$  and  $V_{ik}^{(1)} = \alpha(x_i x_k)/r^3$  do not have bound states (they are repulsive in nature). The wave functions in the potential  $V_{ik}^{(1)} = -\alpha(x_i x_k)/r^3$  are expressed explicitly in terms of the hypergeometric functions. These equations are of interest in investigating the quantum problem of several bodies with Coulomb interaction.<sup>5</sup>

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<sup>1</sup>In a slightly different form, the method is known as *Darboux's transformation*.<sup>2</sup>

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