From Virasoro constraints in Kontsevich's model to *W* constraints in asymmetric 2-matrix model

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The Ward identities in Kontsevich-like 1-matrix models are used to prove at the level of discrete matrix models the suggestion of Gava and Narain's hypothesis, which relates the degree of the potential in the 2-matrix model to the form of the *W* constraints imposed on its partition function.

1. Matrix models, which were originally developed[1] as an alternative approach to (so far two-dimensional) quantum gravity, are now understood to possess a deep and interesting mathematical structure of their own. Their partition functions are usually subjected to infinite sets of constraints, which can be formulated in the form of differential equations with respect to the "time"-variables, and as a corollary they appear to be proportional to the "τ functions" of integrable hierarchies.[2]

While this general scheme is generally well understood at the level of discrete 1-matrix models,[3,4] it is still in many respects an open question when the continuum limit and/or multimatrix modes are considered. An important breakthrough in the study of the continuum limit of 1-matrix models is attributed to a recent Kontsevich's hypothesis.[5] He suggested that the proper continuum limit of the Hermitian 1-matrix model can be described in terms of a slightly different matrix model, below referred to as Kontsevich's model. Its partition function is essentially proportional to the $N\times N$ (anti-)Hermitian matrix integral

$$\mathcal{F}\{\Lambda\} \equiv \int DX \ \exp(-\mathrm{tr}W[X] + \mathrm{tr}\Lambda X), \tag{1}$$

where $W[X] = X^3$, and $N \to \infty$. This hypothesis was strongly supported by a recent proof[6] that the partition function of Kontsevich's model satisfies exactly the same set of differential equations (Virasoro constraints[2]) as the continuum limit of the original Hermitian 1-matrix model (so that it remains only to understand what are the requirements that guarantee the uniqueness of the solution of these Virasoro constraints). In Ref. 1 it was shown that Virasoro constrints arise from the obvious Ward identities, which are satisfied by $\mathcal{F}(\Lambda)$ from (1),

$$(\mathrm{tr}\Lambda^p W'[\partial/\partial\Lambda_{\mathrm{tr}}] + \mathrm{tr}\Lambda^{p+1})\mathcal{F}\{\Lambda\} = 0 \tag{2}$$

as a result of a change (Miwa transformation) of the argument

$$\Lambda \to \{T_m = \frac{1}{m+\frac{1}{2}}\mathrm{tr}\Lambda^{-m-\frac{1}{2}} + \frac{4}{3\sqrt{3}}\delta_{m,1}\}. \tag{3}$$

In turn, Eq. (2) states that integral (1) is invariant under infinitesimal shift of the integration variable,

$$X \to X + \Lambda^p. \tag{4}$$

The implication of Kontsevich's hypothesis in this context (for more physical implication see Ref. 6) is that the Miwa transformation (2) allows us to substitute a sophisticated double-scaling limit in conventional matrix models[1] by a naive limit of $N \to \infty$ in a model like (1). This opens some promising possibilities in the study of the continuum matrix models.

Somewhat unexpectedly, this fresh view of continuum limits provides also a new approach to the study of discrete multimatrix models. One of the main problems about them is the lack of understanding (and even a derivation) of differential equations which substitute Virasoro constraints of the 1-matrix models (and are believed[2] to be expressible in terms of generators of $\mathcal{W}$ algebras). We are going to demonstrate in this letter that the same Ward identities (2) for the matrix integral (1) provide a simple proof of the $\mathcal{W}$-like constraints in a discrete (i.e., at finite N) 2-matrix model.

Moreover, the recent observation due to Gava and Narain,[7] which states that the spin of the $\mathcal{W}$ constraint coincides with the power of the potential $W[X]$, is naturally explained in this manner. (This result was obtained in Ref. 7 by a thorough examination of the continuum limit of loop equations for the specific 2-matrix model with $W[X] = X^3$.)

2. The partition function of the discrete Hermitian 2-matrix model[8] is given by a double integral over $N \times N$ Hermitian matrices X and Λ:

$$Z_{V,W} = \int DX\, D\Lambda \exp(-\mathrm{tr}\{V[\Lambda] + W[X] + \Lambda X\}) \equiv \int D\Lambda\, e^{-\mathrm{tr}V[\Lambda]}\mathcal{F}_W\{\Lambda\}. \tag{5}$$

The potentials V and W are conventionally parameterized by the corresponding time variables

$$V[\Lambda]=\sum_{k\geq 0}^{\infty} t_k\Lambda^k,\quad W[X]=\sum_{k\geq 0}^{K} s_k X^k, \tag{6}$$

and the partition function $Z_{V,W}$ is usually treated as a functional of $\{t_k\}$ and $\{s_k\}$. Below we will use the obvious notation $Z_{V,W}\equiv Z_W\{t_k\}$. While in Ref. 2 it was suggested that the continuum (i.e., $N\to\infty$) limit of $Z_{V,W}$ in the case $W=V$ ($K=\infty$) is annihilated by a set of operators which form $\mathscr{W}_3$ algebra, the results of Ref. 7 imply that the structure of constraints in the asymmetric case, $W\neq V$, is more complicated: The generators of $\mathscr{W}_K$ (expressed in terms of t variables) annihilate $Z_W\{t_k\}$ when $W[X]$ is a polynomial of power K. Our purpose below is to explore the origin of this important phenomenon in the most obvious way.

The natural derivation arises from comparison of Eqs. (5) and (2). Indeed,

$$\frac{\partial Z_W}{\partial t_{p+1}}=\int D\Lambda\, e^{-\mathrm{tr}V[\Lambda]}\mathrm{tr}\Lambda^{p+1}\mathcal{F}\{\Lambda\}=\int D\Lambda\, e^{-\mathrm{tr}V[\Lambda]}\mathrm{tr}(\Lambda^p W'[\frac{\partial}{\partial\Lambda_{\mathrm{tr}}}])\mathcal{F}\{\Lambda\}. \tag{7}$$

After integration by parts this equations becomes

$$\int D\Lambda\mathcal{F}\{\Lambda\}\mathrm{tr}(W'[-\frac{\partial}{\partial\Lambda_{tr}}]\Lambda^p)e^{-\mathrm{tr}V[\Lambda]}$$
$$=\sum_{k>0}^{K} ks_k\int D\Lambda\mathcal{F}\{\Lambda\}\mathrm{tr}((-\frac{\partial}{\partial\Lambda_{\mathrm{tr}}})^{k-1}\Lambda^p)e^{-\mathrm{tr}V[\Lambda]}. \tag{8}$$

The leading term in the sum on the r.h.s., i.e.,

$$Ks_K\int D\Lambda\mathcal{F}\{\Lambda\}\{\mathrm{tr}\,\Lambda^p(V'[\Lambda])^{K-1}+O(V^{K-2})\}$$
$$=Ks_K\sum_{a_1,\dots,a_{K-1}} a_1 t_{a_1\dots a_{K-1}}t_{a_{K-1}}\frac{\partial}{\partial t_{a_1+\dots+a_{K-1}+p+1-K}}Z_W\{t\}, \tag{9}$$

is a "classical"[1)] piece of the operator $\mathscr{W}^{(K)}_{p+1-K}$ – $(p+K-1)$-th harmonic of the spin-K generator of $\mathscr{W}_K$ algebra which acts on $Z_W\{t\}$. [Note that according to (8), $p\geqslant 0$.] This essentially explains the origin of the $\mathscr{W}_K$ constraint and its intimate relation to the form of the potential $W[\Lambda\}$, in complete agreement with Ref. 7.

In general, the Ward identity (7) can be rewritten as a set of constraints:

$$\left(-\frac{\partial}{\partial t_{p+1}}+\sum_{k>0}^{K} ks_k\tilde{W}^{(k)}_{p+1-k}\{t\}\right)Z_W\{t\}=0. \tag{10}$$

The operators $\tilde{\mathscr{W}}^{(k)}$ are defined by

$$\tilde{W}^{(k)}_{p+1-k}e^{-\mathrm{tr}V[\Lambda]} = \mathrm{tr}((-\frac{\partial}{\partial\Lambda_{\mathrm{tr}}})^{k-1}\Lambda^p)e^{-\mathrm{tr}V[\Lambda]}. \tag{11}$$

They obey the recurrence relation

$$\tilde{W}^{(k+1)}_p = \sum_n nt_n \tilde{W}^{(k)}_{n+p} + \sum_{a+b=p+k-1} \frac{\partial}{\partial t_a}\tilde{W}^{(k)}_{b+1-k}; \quad p \geq -k \tag{12}$$

with

$$\tilde{W}^{(2)}_p = \mathcal{L}_p = \sum_n nt_n \frac{\partial}{\partial t_{n+p}} + \sum_{a+b=p} \frac{\partial^2}{\partial t_a \partial t_b}; \quad p \geq -1, \tag{13}$$

$$\tilde{W}^{(1)}_p = J_p = \frac{\partial}{\partial t_p}; \quad p \geq 0. \tag{14}$$

Equations (12) and (13) imply that all the $\tilde{\mathscr{W}}^{(k)}$ operators are in fact proportional to the linear combinations of the Virasoro opertors $\mathscr{L} = \tilde{\mathscr{W}}^{(2)}$. This may explain how in the continuum limit a single Ward identity (10) can give rise to the entire set of constraints with lower spins. However, this topic is beyond the scope of this letter. The continuum limit of these equations may be studied along the lines of Ref. 7.

3. While from the point of view of derivations it is illuminating (first) to study the simplest Ward identity (3) in the "1-matrix component" of a 2-matrix model, and then to apply it to the (more sophisticated) analysis of a 2-matrix case, it is of course possible to treat the resulting $\mathscr{W}$ constraints (10) as Ward identities in the entire 2-matrix model, which is related to the following infinitesimal change of the integration variables

$$\delta X = \Lambda^p, \quad p \geq 0 \tag{15}$$

$\delta\Delta = (\Sigma^K_{m=0} ms_m \Sigma^{m-2}_{k=0}(-)^{k+1}(V')^{k+1}X^{m-2-k})\Lambda^p +$ "quantum corrections."
This variation of variables induces the variation of the potential:

$$\delta S = \left(\sum_{m=0}^{K} ms_m(-)^m(V')^{m-1}\right)\Lambda^p + \Lambda^{p+1} + \text{"quantum corrections"}. \tag{16}$$

The first term in this expression gives rise to the "classical" part of $\tilde{\mathscr{W}}$ algebra and the second term produces the derivative $\partial/\partial t_{p+1}$ in (10).

While the X component of the variation (15) [which is Eq. (3)] does not change the integration measure $DXD\Lambda$, this is not true for the Λ component. The corresponding Jacobian is responsible for the "quantum" contributions to (15) and (16).

Equations (15) and (16) give a possible generalization of the 2-matrix case of the derivation[9,10] of the Virasoro constraints in the discrete 1-matrix model from the Ward identities associated with the shift $M \to M + \epsilon M^{n+1} (n \geqslant -1)$ of the integration variable.

4. Finally, we have several comments.

First of all, we should stress that Eq. (11) defines only positive ($p \geqslant 1-K$, to be exact) harmonics $\tilde{\mathscr{W}}_p^{(K)}$ of the $\tilde{\mathscr{W}}^{(K)}$ operators.

Second, an important question is whether the set of $\tilde{\mathscr{W}}$ constraints (10) is closed. It is indeed the case. Specifically, it can be shown that

$$[\tilde{\mathcal{W}}_p^{(K)}, \tilde{\mathcal{W}}_q^{(K)}] \in Span\, \tilde{\mathcal{W}}_r^{(K)} \tag{17}$$

and $r \geqslant 1-K$ as long as $p,q \geqslant 1-K$. In particular,

$$[\tilde{\mathcal{W}}_p^{(2)}, \tilde{\mathcal{W}}_q^{(2)}] = (p-q)\tilde{\mathcal{W}}_{p+q}^{(2)}, \; p,q \geq -1, \tag{18}$$

$$[\tilde{\mathcal{W}}_p^{(3)}, \tilde{\mathcal{W}}_q^{(3)}] = 2(p-q)\sum_k kt_k \tilde{\mathcal{W}}_{p+q+k}^{(3)}$$

$$+ \left[\sum_{a=0}^{p+1}(2p-q-2a) - \sum_{a=0}^{q+1}(2q-p-2a)\right] \frac{\partial}{\partial t_b} \tilde{\mathcal{W}}_{p+q-a}^{(3)}, \; p,q \geq -2, \tag{19}$$

etc.

It should be noted that (17) remains a highly nontrivial property of $\tilde{\mathscr{W}}$ algebra: The fact that this set is closed makes an interesting exercise for observing the appearance of the entire set of $\tilde{\mathscr{W}}^{(n)}$ constraints (with all spins $n \leqslant K$) in the continuum limit from the single spin-K constraint at the discrete level.

Third, the same $\tilde{\mathscr{W}}$ operators were found in Ref. 6 in a somewhat different context. It can be proved that they are really the same, thus demonstrating a kind of universal nature of $\tilde{\mathscr{W}}$ operators, at least in the framework of discrete matrix models.

Fourth, the fact that the commutator of $\tilde{\mathscr{W}}^{(K)}$ operators in (17) is proportional not only to $\tilde{\mathscr{W}}_{p+q}^{(2K-2)}$ demonstrates that $\oplus_K \tilde{\mathscr{W}}^{(K)}$ is not a Lie algebra (to make it similar to $\mathscr{W}_\infty$, at least the basis should be changed). This makes $\tilde{\mathscr{W}}$ even more similar to the conventional $\mathscr{W}$ algebras, which are also nonlinear and closed when only the operators $\tilde{\mathscr{W}}^{(n)}$ of spins $n \leqslant K$ are considered.

5. In summary, we demonstrated that the Ward-identities of Kontsevich-like models, which are derived in Ref. 6, are sufficient to obtain a closed (and presumably) complete set of Ward identities (loop equations) in the discrete 2-Hermitian matrix model. These loop equations involve $\tilde{\mathscr{W}}$ operators acting on one of the potentials in the 2-matrix model, while the "highest" spin found in these operators coincides with the power of another potential. One can therefore view these constraints as Ward identities of the proper 1-matrix model (in terms of the variable Λ)[2)] with a suitable measure produced by integration over matrix X in (5). In principle, it is a rather

trivial statement that there exists an algebra of $\widetilde{\mathscr{W}}$ constraints in any 1-matrix model, since it can be produced by conjugation from evident algebra (more precisely, Borel subalgebra) of the operators which cancel the identity partition function.[12,13] This statement is the discrete counterpart of Gava and Narain[1,7] suggestion concerning the "asymmetric" (i.e., two potentials do not coincide) continuum limit of the 2-matrix model, which is probably different from the symmetric limit orginally examined in Ref. 2.

At the discrete level the $\widetilde{\mathscr{W}}$ operators are proportional to the Virasoro operators, so that any solution of the discrete Virasoro constraints automatically satisfies $\widetilde{\mathscr{W}}_p^{(K)}Z = 0$ with $p \geqslant 1 - K$. The proper Ward identity in the 2-matrix model, however, is a linear combination of $\widetilde{\mathscr{W}}_p^{(K)}$ and $\partial/\partial t_{p+K}$ with nonvanishing coefficients, so that the actual constraints imposed on the partition function of the 2-matrix model cannot be expressed in terms of the Virasoro generators.

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1) In the sense of Ref. 9.

2) In contrast with the "true" 2-matrix model, the $\mathscr{W}$ constraints depend on the two sets of times; see the example in Ref. 11.

3) We know of the attempt to construct such constraints in a more explicit way.[13] Unfortunately, it was done in the space of the spectral parameter in terms of Baker-Akhiezer functions, instead of the more transparent language of the partition (τ) functions which depend on the time variables. We hope that our approach with the second potential truncated is more revealing.

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