# Theory of the radial-orbit instability as a universal cause of structure formation in stellar systems

V. L. Polyachenko
*Institute of Astronomy, Academy of Sciences of the USSR, 109017, Moscow*

(Submitted 21 October 1991)
Pis'ma Zh. Eksp. Teor. Fiz. **54**, No. 10, 553–556 (25 November 1991)

An analytic theory is derived for a basic instability of collisionless gravitating systems: the radial-orbit instability. Radial orbits arise in a natural way upon (for example) the formation of galaxies during the collapse of a stellar cloud from an initial state with an extremely low density. The theory shows that the instability occurs if the gravitational potential of the system leads to a precession of orbits with small angular momenta in the direction in which the stars are revolving in these orbits.

A collisionless collapse of a cloud of stars starting from a highly nonequilibrium state is presently regarded as a major factor in possible scenarios for the formation of various stellar systems. For example, it has been shown in several numerical $N$-body experiments[1–4] that elliptical galaxies probably form this way. Only in those cases in which the initial state of the collapsing system is sufficiently "cold" (having a virial ratio $V=2T/|W|\ll 1$, where $T$ is the initial kinetic energy of the system, and $W$ is the initial potential energy; at equilibrium we would have $V=1$) do the "experimental" density distributions found as a result superimpose well on the observed distributions[1)] and therefore correspond to reality. In the course of such collapses, however, a great preponderance of the gravitational energy released in the course of the compression would go into radial motion of the constituents of the system of stars: Orbits which are highly prolate along radius would become predominant. This circumstance would lead in turn to an instability (which is today called the "radial-orbit instability"): By heating the system in the transverse direction (transverse with respect to the radius), this instability would reduce the anisotropy in the distribution of stars with respect to radial and transverse velocities to a certain critical level. This instability, like most others which tend to equalize temperatures in a gravitating medium (i.e.,. which tend to make the medium more nearly isotropic), is basically of a Jeans nature.[6] We recall that the classical Jeans instability of a gaseous medium is simply the gravitational contraction of volumes which are massive enough that pressure forces are no longer capable of withstanding gravitation (see Ref. 7, for example, for the details). Clearly, the cooler the medium, the shorter the wavelengths at which perturbations unstable "in the Jeans sense" begin, i.e., the less stable the medium.

We have a corresponding situation in the case at hand: If we imagine a system with radial orbits, and if we move these orbits closer together in some cone at a certain time, then a subsequent closing of these orbits on each other (i.e., the radial-orbit instability) would be completely natural, since the system is cold in the transverse direction. Actually, as we will see below, this explanation is not complete: Whether an instability occurs depends on not only the fact that there are radial orbits but also on the nature of the precessional motions of the orbits which arise upon perturbations. An instability develops only in the case of a forward precession, i.e., a precession in the direction in which the stars are revolving in their orbits. In the case of a retrograde precession, the instability does not occur. However, in those cases in which the instability does occur, it is clearly of a Jeans nature. The only distinguishing feature of this case is that it would be more accurate to identify the orbits as a whole rather than the individual stars as the elementary entities involved here. In terms of its importance for collisionless gravitating systems, the radial-orbit instability is also completely comparable to the ordinary Jeans instability for a gaseous medium. Collisionless systems are at least as common as gaseous systems in the realm of astronomical objects.

The radial-orbit, instability was first pointed out in Ref. 8. It was later discovered in direct $N$-body experiments[1,4] and through numerical solution of a linearized kinetic equation.[9] In the latter study, and in subsequent papers by the author (see Ref. 10 for the details), corresponding conditions for the stopping of this instability were also found. In other words, the minimum kinetic energy of the transverse motion of the stars sufficient to stop the instability was found.

Antonov[11] attempted to analytically prove the existence of an instability in systems with radial orbits. However, the conclusions which he drew—that any system with a purely radial motion of stars is automatically unstable and that such systems have no stable modes—are incorrect, as we have already mentioned. For example, if the precession of orbits is retrograde, the instability is replaced by a purely oscillatory (i.e., stable) mode. The only way to derive the correct result is to start with a system in which the orbits are highly prolate but not purely radial and then take the limit of a purely radial motion. If we attempt instead to work with the limiting system from the outset, we run into meaningless integrals (which diverge as $r \to 0$).

Since the nature of the radial-orbit instability obviously does not depend on the particular form of the system, we will carry out our calculations for the simple case of cylindrical geometry. Working from a linearized kinetic equation in terms of action-angle variables [respectively $\vec{I} = (I_1, I_2)$ and $\vec{w} = (w_1, w_2)$] in its usual form, we can put this equation in the following form bymeans of the substitutions $\bar{w}_2 = w_2 - w_1/2$ and $\bar{w}_1 = w_1$:

$$\frac{\partial F}{\partial t} + im\Omega_{pr} F + \Omega_1 \frac{\partial F}{\partial w_1} = \Omega_1 \frac{\partial F_0}{\partial E} \frac{\partial \Phi}{\partial w_1} + im\Phi \left( \frac{\partial F_0}{\partial L} + \Omega_{pr} \frac{\partial F_0}{\partial E} \right). \qquad (1)$$

Here $F_0(E,L) = f_0(I_1, I_2)$ and $F$ are the unperturbed and perturbed distribution functions; $\Phi$ is the perturbation of the potential [ $\propto \exp(im\bar{w}_2)$, where $m$ is an integer]; $\Omega_1$ and $\Omega_2$ are the frequencies of the radial and azimuthal oscillations of the star in the equilibrium potential $\Phi_0(r)$; and $\Omega_{pr}(E,L) = \Omega_2 - \Omega_1/2$ is the precession velocity of an orbit with an energy $E$ and an angular momentum $L$. We assume that the spread in precession velocity, $\delta\Omega_{pr} = (\overline{\Omega_{pr}^2})^{1/2}$, and the characteristic Jeans frequency $\omega_J = \sqrt{4\pi G\bar{\rho}_0}$ ($\rho_0$ is the density, and $G$ the gravitational constant) are both small: $\delta\Omega_{pr}, \omega_J \ll \Omega_1$. This assumption means that we are dealing with a system of stars with nearly radial orbits inside a massive halo, which basically determines the potential $\Phi_0$

(without participating in perturbations). Under such conditions there exists a low-frequency mode [ $\propto \exp(-i\omega t)$ with $\omega \sim \omega_J, \delta\Omega_{pr}$] in which a slow precessional dispersal of the orbits is offset by their mutual gravitational attraction.

We seek a solution by perturbation theory, writing $F = F^{(1)} + F^{(2)}$, where $F^{(1)}$ corresponds to the "exchange" mode, which can be found from (1) by ignoring the terms proportional to $\Omega_{pr}$ and to $\Phi \propto G$: $\omega = 0$, $\partial F^{(1)}/\partial \omega_1$. In other words, $F^{(1)} = F^{(1)}(E,L)$ is a function of the integrals of motion (an arbitrary function at this point; below we will find a more concrete expression for this function by working from the periodicity of the solution of the next approximation, $F^{(2)}$). The equation for $F^{(2)}$ is

$$-i\omega F^{(1)} + im\Omega_{pr} F^{(1)} + \Omega_1 \frac{\partial F^{(2)}}{\partial w_1} = \Omega_1 \frac{\partial F_0}{\partial E}\frac{\partial \Phi}{\partial w_1} + im\Phi \left(\frac{\partial F_0}{\partial L} + \Omega_{pr}\frac{\partial F_0}{\partial E}\right).$$

Integrating over $\omega_1$ from 0 to $2\pi$, taking the periodicity of the functions $F^{(2)}$ and $\Phi$ into account, and (for simplicity) restricting the discussion to modes which are small-scale modes in terms of angle, with $m \gg 1$, we find the following result:

$$-(\omega - m\Omega_{pr}^{(1)})F \simeq m \left(\frac{\partial F_0}{\partial L} + \Omega_{pr}\frac{\partial F_0}{\partial E}\right)\frac{1}{2\pi}\int_0^{2\pi} \Phi dw_1. \tag{2}$$

Invoking the Poisson equation, making use of the small quantity $(\omega - m\Omega_{pr})$, taking the limit of radial orbits, and carrying out after some manipulations, we find the following integral equation for the function $\chi(E) = 1/2\pi \int_0^{2\pi} \Phi d\omega_1$:

$$-\omega^2 \chi(E) = \int dE_1 2K(E, E_1)\chi(E_1), \tag{3}$$

with the kernel $[(F_0 = \delta(L)\varphi_0(E)]$

$$K(E, E_1) = 4\pi G\left(\frac{\partial \Omega_{pr}(E_1, L)}{\partial L}\right)_{L=0} \cdot \varphi_0(E_1)\int \frac{r\,\partial r}{\sqrt{E_1 - \Phi_0(r)}\sqrt{E - \Phi_0(r)}}.$$

This equation has an eigenvalue $(-\omega^2) \sim 4\pi G\bar{\rho}_0 R^2 (\overline{\partial\Omega_{pr}/\partial L})_{L=0}$, where $R$ is the size of the system, for a mode with no nodes along $E$. All the assertions which we made above then follow. An instability ($\omega^2 < 0$) occurs if the orbits precess in the forward sense, and the relation $(\partial\Omega_{pr}/\partial L)_{L=0} > 0$ holds. In particular, for a potential $\Phi_0 = \Omega^2 r^2/2 + br^4$, an instability occurs if $b < 0$, while in the case $b > 0$ we have pure oscillations instead of an instability. The potential $\Phi_0$ is of this nature in the central region (if there is no point mass there). However, it is the central regions which play the leading role for gravitating systems, primarily because the density of matter is highest in these regions. On the basis of this result we can (for example) classify galaxies on the basis of the behavior of the gravitational potential near the center. A structure which disrupts the original symmetry (such as a central elliptical bar in a spiral galaxy) forms if the profile of the potential is smoother than quadratic.

[1)]The surface brightness $I(R)$ [and thus the surface density $\sigma(R)$] of elliptical galaxies is described well by the universal de Vaucouleurs law[5] $I(R) \propto \exp[-7.67(R/R_e)]^{1/4}$, where $R_e$ is the radius which bounds half the emitted light, and which does not depend on the size and mass of the galaxy.

[1]B. L. Polyachenko, Pis'ma Astron. Zh. 7, 142 (1981) [Sov. Astron. Lett. **7**, 79 (1981)].
[2]T. S. Van Albada, Mon. Not. R. Astron. Soc. **201**, 939 (1982).
[3]T. A. Mclynn, Astrophys. J. **281**, 13 (1984).
[4]V. L. Polyachenko, Astron. Tsirk., No. 1405, 4 (1985).
[5]G. H. de Vaucouleurs, *Structure of Stellar Systems* [Russian translation], IL, Moscow, 1962.
[6]V. L. Polyachenko and A. M. Fridman, Zh. Eksp. Teor. Fiz. **94**(5), 1 (1988) [Sov. Phys. JETP **67**, 867 (1988)].
[7]Ya. B. Zel'dovich and I. D. Novikov, *The Structure and Evolution of the Universe,* U. Chicago P., Chicago, 1983.
[8]V. L. Polyachenko and I. G. Shukhman, Preprint 1-2, Siberian Institute of Terrestrial Magnetism, the Ionosphere, and Radio Wave Propagation, Academy of Sciences of the USSR, Irkutsk, 1972.
[9]V. L. Polyachenko and I. G. Shukhman, Astron. Zh. **58**, 933 (1981) [Sov. Astron. **25**, 533 (1981)].
[10]A. M. Fridman and V. L. Polyachenko, *Physics of Gravitating Systems, Vol. 1,* Springer-Verlag, New York, 1984.
[11]V. A. Antonov, *Dynamics of Galaxies and Star Clusters,* Alma-Ata, 1973, p. 139.

Translated by D. Parsons