Nonrenormalization theorem in Chern–Simons theories and the fractional quantum Hall effect

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A nonrenormalization theorem is proved. It leads to the conclusion that a superconductivity exists in a system of free and interacting anyons. The theorem is used to prove that a quantization of the transverse conductivity occurs in a proposed new formulation of the theory of the fractional quantum Hall effect.

Two-dimensional systems of particles with fractional statistics (anyons) have recently attracted widespread interest in connection with the problem of high-T_c superconductivity and the fractional quantum Hall effect. It has been shown in the random phase approximation, in a system of many particles with a fractional statistics which can be described by the parameter $1/N$ (the value $N=1$ corresponds to a conversion of fermions into bosons), that there exist (a) a gapless collective excitation which corresponds to oscillations of the density, (b) a Meissner effect, and (c) a superconductivity.[1–3] In a formulation as a theory with a Chern–Simons term, the presence of a massless excitation in the spectrum means the vanishing of a corresponding coefficient in the effective Lagrangian for the statistical gauge field found after an integration over the fermion field (the initial and induced topological terms cancel out).[3] Coleman and Hill[4] (see also Ref. 5) demonstrated that for relativistically invariant theories with massive fields of matter there is only a single-loop contribution to the coefficient of the induced Chern–Simons term. Lykken *et al.*[6] showed, by analogy with Ref. 4, that the Chern–Simons term in a system of massive Dirac fermions cancels out at a finite density, at a certain value of the coefficient of the seed topological term. They did not, however, examine the possible existence of infrared divergences, which might alter the simple calculation of the powers of the external momentum.

Coleman and Hill's arguments[4] make it possible to demonstrate, without the use of the random phase approximation, that a gapless mode exists in a system of nonrelativistic particles with a fractional statistics ($1/N$). The idea here is that, after the average magnetic field is singled out, the fermion propagators in the new diagram technique are the Green's functions of electrons in a system with N completely filled Landau levels. In other words, they have a finite gap. It follows from the gauge invariance and the analyticity of the effective phonon vertices at zero momentum that only the single-loop diagram contributes to the renormalization of the Chern–Simons term. An important point here is that the effective n-photon vertex is of order $O(p_1,\ldots,p_n)$ in the external momenta p_i. In proving this fact, Coleman and Hill[4] made use of the relativistic invariance, but that procedure is not obligatory. The reason is that for $n>2$ the vertex may be formed exclusively from gauge-invariant combinations of f_{0i} and f_{ij} ($f_{\mu\nu}(a) = \partial_\mu a_\nu - \partial_\mu a_\nu$, since the combination $\epsilon^{\mu\nu\alpha}a_\mu f_{\nu\alpha}$ changes by a total derivative and can arise only for $n=2$. Consequently, there are no infrared divergences in the integration over the momenta of the internal photon lines, and the simple calculation of the power of the external momentum in Ref. 4 is meaningful. We show below that a massless mode also exists in a system of interacting anyons. The interaction, which we assume to be a binary interaction and independent of the velocity, is incorporated in a diagram technique through the substitution $a_0 \to a_0 + \alpha$, where the propagator of the field α is $V(\vec{p})$. If the Fourier transform $V(\vec{p})$ of the interaction potential $V(\vec{x})$ has no singularities at small $\vec{p}$, the arguments of Ref. 4 apply without change: Diagrams with α lines do not contribute to the coefficient of the Chern-Simons term. If $V(\vec{p}) \sim 1/|\vec{p}|$, then a vertex of this sort is proportional to the momentum $\vec{p}$ corresponding to the α line and could not lead to infrared divergences, since the field α appears in effective vertices with $n>2$ (n is the total number of a and α lines) only in the combination $f_{0i}(a_0+\alpha, \vec{a})$. A vertex with $n=2$ and two α lines contributes to the complete α propagator and cannot change its behavior $\propto 1/|\vec{p}|$. Nor can a diagram with $n=2$ and one α line alter the behavior of the α and a_μ propagators. We thus obtain a diagram for the polarization operator $\Pi_{\mu\nu}(p)$ with two $n=2$ vertices connected by a single α line, which is $O(p^2)$ [for example, component $\Pi_0 i(p)$ is proportional to $\vec{p}^2 V(\vec{p})\epsilon_{ij}p_j$] and cannot contribute to a renormalization of the Chern–Simons term. The theorem has thus been proved, at least for potentials which fall off no more slowly than $1/|\vec{x}|$ at large distances (a three-dimensional Coulomb potential).

Let us apply the formulation of the theory with a topological term and the nonrenormalization theorem which we have proved to the case of the fractional quantum Hall effect. We consider a system of two-dimensional electrons in a transverse magnetic field with a filling factor $\nu = 1/m$, where m is odd. Following Ref. 7, we transform the wave function ($z_i = x_i + iy_i$):

$$\Psi'(\vec{x}_1,\ldots\vec{x}_N) = \prod_{i<j}\left(\frac{(z_i - z_j)}{|z_i - z_j|}\right)^{m-1} \Psi(\vec{x}_1,\ldots\vec{x}_N). \tag{1}$$

The new wave function is also antisymmetric, i.e., also corresponds to Fermi statistics. Going over to second quantization, and introducing the gauge field a_μ, we obtain the Lagrangian

$$L = i\psi^+(\partial_0 - iA_0 - ia_0)\psi - a_0\rho_0$$

$$+\frac{1}{2M}\psi^+(\partial - i\vec{A}_1 - i\vec{A} - i\vec{a})^2\psi + \frac{1}{4\pi(m-1)}\epsilon^{\mu\nu\alpha}a_\mu\partial_\nu a_\alpha, \tag{2}$$

where the field ψ obeys Fermi statistics, ρ_0 is the density of electrons, $\mathbf{A}_1$ is the vector potential corresponding to the uniform magnetic field $H = 2\pi\rho_0$, in which the electrons completely fill one Landau level, and A_μ is the external electromagnetic potential. We need to add an electron-electron interaction to Lagrangian (2). This added interaction lifts the strong degeneracy of the ground state. The theory which we have constructed, (2), is completely equivalent to the original theory, but it has several features which are characteristic of an effective theory (a Ginzburg–Landau theory)[8] [the transformation of (1) to a boson wave function was discussed in Ref. 9]. For example, quasiparticles with a fractional charge[10] correspond to vortices which acquire a finite energy because of the Coulomb interaction. Fluctuations in the system of vortices and antivortices should correspond to a magnetophonon mode (fluctuations in the lowest Landau level).[11] To demonstrate this point, we use the substitution $a_0 \to a_0 - \alpha$ and integrate over the field α. The interaction can then be written

$$\int d\vec{x} d\vec{y} (\epsilon_{ij}\partial_i a_j)(\vec{x}) V(\vec{x}-\vec{y})(\epsilon_{ij}\partial_i a_j)(\vec{y}).$$

This interaction should introduce an energy gap in the quasiparticle excitations, because of the nonzero statistical magnetic field at the center of a vortex. In this formulation, we can single out an effective Lagrangian which describes the system of vortices. The proposed formulation is a realization of Jain's idea[7] of the binding of an even number of flux quanta (fluxoids) by electrons and of an integer quantum Hall effect for composite entities. This formulation may prove convenient for reaching a better understanding of the incompressibility of states with $\nu = 1/m$. The nonrenormalization theorem which we have proved is being used in this case to explain the exact quantization of the transverse conductivity in a system of interacting electrons.

We integrate over the fermion fields in a functional integral of (2). The effective single-loop Lagrangian for slowly varying fields a_μ and A_μ is[12]

$$L_{eff}(a,A) = \frac{1}{4\pi}\left(\frac{M}{H}\vec{E}^2(a+A) - \frac{1}{M}H^2(a+A)\right)$$
$$+\frac{1}{4\pi}\epsilon^{\mu\nu\alpha}(a+A)_\mu\partial_\nu(a+A)_\alpha + \frac{1}{4\pi(m-1)}\epsilon^{\mu\nu\alpha}a_\mu\partial_\nu a_\alpha, \tag{3}$$

where $\vec{E}$ and H are the electric and magnetic fields ($E_i(a) = f_{0i}(a)$, $H(a) = \epsilon_{ij}\partial_i a_j$). By virtue of the theorem which we have proved, only the first two terms in (3) can contain corrections for higher loops; such corrections do not contain an induced topological term. The Chern–Simons term (for a_μ) does not cancel out; this situation corresponds to a gap in the spectrum of collective excitations. The dispersion relation for a magnetoplasmon excitation[11] corresponding to the field a_μ is

$$\omega(\vec{p}) = \frac{m}{m-1}\frac{H}{M}\left[1 + \frac{\vec{p}^2(m-1)^2}{m^2}(1/H)\right]^{1/2} \tag{4}$$

Expression (4) was derived in the single-loop approximation. To calculate the transverse conductivity, we integrate over the statistical gauge field a_μ in (3). For the term with the smallest number of derivatives in the action for the external electromagnetic field, it is sufficient to consider the last two terms in (3):

$$S_{eff}(A) = \int d^3x \frac{1}{4\pi m}\epsilon^{\mu\nu\alpha}A_\mu\partial_\nu A_\alpha. \tag{5}$$

This result corresponds to the known value $\delta_{xy} = 1/(2\pi m)$. This result is exact by virtue of the nonrenormalization theorem. We have also proved that incorporating the electron–electron interaction does not change the quantity in (5). This analysis can easily be generalized to filling factors $\nu = p/(2kp \pm 1)$ (where $\vec{p}$ and $\vec{k}$ are integers) in the first level of the hierarchy of incompressible states,[7] for which the $\vec{p}$ Landau levels are completely filled after a transformation of wave function (1) with $m = 2k \pm 1$.

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