A class of solutions of the steady-state vacuum Einstein equations

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A new class of steady-state axisymmetric solutions of the Einstein equations is constructed.

Hoenselaers *et al.*[1] and Yamazaki[2] have demonstrated the existence of an axisymmetric solution (other than the Kerr solution) of the steady-state vacuum Einstein equations. That solution apparently also describes the gravitational field of a rotating mass.

In the present letter we derive a class of solutions of the steady-state vacuum Einstein equations. In a particular case, this class of solutions becomes the Hoenselaers–Yamazaki solution.[1,2]

The metric of a steady-state, axisymmetric gravitational field in Weyl coordinates is known to be

$$dS^2 = f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2] - f(dt - \omega d\varphi)^2, \tag{1}$$

where the unknown functions f, ω, and γ depend on only the coordinates ρ and z. It is customary to also introduce a function $\Phi(\rho,z)$, which is related to f and ω by

$$\Phi_{,\rho} = \rho^{-1} f^2 \omega_{,z}, \qquad \Phi_{,z} = -\rho^{-1} f^2 \omega_{,\rho}. \tag{2}$$

(Here and below, a subscript comma means the operation of partial differentiation.)

In this case the vacuum Einstein equations can be written for the single complex function ϵ,

$$\epsilon = f + i\Phi, \tag{3}$$

in Ernst form:

$$(\epsilon + \epsilon^*)(\epsilon_{,\rho\rho} + \epsilon_{,zz} + \rho^{-1}\epsilon_{,\rho}) = 2(\epsilon_{,\rho}^2 + \epsilon_{,z}^2). \tag{4}$$

The metric coefficient γ can be found if ϵ is known.

If ϵ is a solution of Eq. (4), then

$$\tilde{\epsilon} = \epsilon^{-1} \tag{5}$$

is also a solution.

Using the results of Ref. 3, we can write a superposition of a Kerr solution with an arbitrary static vacuum field ψ as follows:

$$\epsilon = e^{\psi} \frac{\alpha Q + \beta P - 2k}{\alpha Q + \beta P + 2k} , \qquad (6)$$

$$\alpha = \sqrt{\rho^2 + (z+k)^2}, \qquad \beta = \sqrt{\rho^2 + (z-k)^2}, \qquad (7)$$

where k is a real constant, and the static field ψ satisfies the equation

$$\psi_{,\rho\rho} + \psi_{,zz} + \rho^{-1}\psi_{,\rho} = 0. \qquad (8)$$

The functions P and Q are defined by

$$P = \frac{1+iA}{1-iA}, \qquad Q = \frac{1+iB}{1-iB}, \qquad (9)$$

where A and B must be determined, for the given static solution ψ, from the following system of first-order differential equations:

$$\begin{aligned} (\ln A)_{,\rho} &= \beta_{,\rho}\psi_{,z} + \beta_{,z}\psi_{,\rho}, \\ (\ln A)_{,z} &= \beta_{,z}\psi_{,z} - \beta_{,\rho}\psi_{,\rho}, \\ (\ln B^{-1})_{,\rho} &= \alpha_{,\rho}\psi_{,z} + \alpha_{,z}\psi_{,\rho}, \\ (\ln B^{-1})_{,z} &= \alpha_{,z}\psi_{,z} - \alpha_{,\rho}\psi_{,\rho}. \end{aligned} \qquad (10)$$

As the static solution ψ we adopt the function

$$\psi = \delta \ln \frac{S_- + \sqrt{\rho^2 + S_-^2}}{S_+ + \sqrt{\rho^2 + S_+^2}}, \qquad (11)$$

$$\begin{aligned} \sqrt{2}S_+ &= [(z+k)^2 + \alpha_+\alpha_- - \rho^2 - \epsilon_0^2]^{1/2}, \\ \sqrt{2}S_- &= [(z-k)^2 + \beta_+\beta_- - \rho^2 - \epsilon_0^2]^{1/2}, \end{aligned} \qquad (12)$$

$$\begin{aligned} \alpha_+ &= \sqrt{\rho^2 + (z+k+\epsilon_0)^2}, \qquad \alpha_- = \sqrt{\rho^2 + (z+k-\epsilon_0)^2}, \\ \beta_+ &= \sqrt{\rho^2 + (z-k+\epsilon_0)^2}, \qquad \beta_- = \sqrt{\rho^2 + (z-k-\epsilon_0)^2}, \end{aligned} \qquad (13)$$

where the constants δ and ϵ_0 characterize the deviation of the field from spherical symmetry. The function in (11) satisfies Eq. (8). It becomes the Schwarzschild solution in the case $\delta = 1/2$, $\epsilon_0 = 0$.

We calculate the derivatives $\psi_{,\rho}$, $\psi_{,z}$, $\beta_{,\rho}$, $\beta_{,z}$, $\alpha_{,\rho}$, and $\alpha_{,z}$ with the help of (7), (11), (12), and (13).

Substituting the results into (10), we find

$$\frac{1}{\delta}(\ln A)_{,\rho}=\frac{(\beta_++\beta_-)[\rho^2-(z-k)^2]+(\beta_+-\beta_-)(z-k)\epsilon_0}{2\rho\beta\beta_+\beta_-}$$

$$-\frac{(\alpha_++\alpha_-)[\rho^2-(z^2-k^2)]+(\alpha_+-\alpha_-)(z+k)\epsilon_0}{2\rho\beta\alpha_+\alpha_-}, \tag{14}$$

$$\frac{1}{\delta}(\ln A)_{,z}=\frac{2(\beta_++\beta_-)(z-k)+(\beta_+-\beta_-)\epsilon_0}{2\beta\beta_+\beta_-}$$
$$-\frac{2(\alpha_++\alpha_-)z-(\alpha_+-\alpha_-)\epsilon_0}{2\beta\alpha_+\alpha_-}. \tag{15}$$

There are corresponding expressions for $(\ln B^{-1})_{,\rho}$ and $(\ln B^{-1})_{,z}$, with β replaced by α, and $\beta_\pm$ by $\alpha_\pm$.

Integrating (14), (15), and the corresponding equations for B, we finally find

$$A=a\left(\frac{KR}{L}\right)^{\delta/2}, \tag{16}$$

$$B=b\left(\frac{MG}{N}\right)^{\delta/2}. \tag{17}$$

$$K=[(\beta+\beta_+)^2-\epsilon_0^2][(\beta+\beta_-)^2-\epsilon_0^2],$$
$$L=[(\beta+\alpha_-)^2-4k(k-\epsilon_0)]^{l_-}[(\beta+\alpha_+)^2-4k(k+\epsilon_0)]^{l_+}, \tag{18}$$

$$R=(\beta+\alpha_+)^{l_+-1}(\beta+\alpha_-)^{l_--1}\rho^{l_++l_--2},$$
$$M=[(\alpha+\alpha_+)^2-\epsilon_0^2][(\alpha+\alpha_-)^2-\epsilon_0^2], \tag{19}$$
$$N=[(\alpha+\beta_-)^2-4k(k-\epsilon_0)]^{l_-}[(\alpha+\beta_+)^2-4k(k+\epsilon_0)]^{l_+},$$
$$G=(\alpha+\beta_+)^{l_+-1}(\alpha+\beta_-)^{l_--1}\rho^{l_++l_--2},$$

where a and b are integration constants, and $l_\pm$ are given by

$$l_\pm=\frac{z+k}{z-k}\;\frac{z-k\pm\epsilon_0}{z+k\pm\epsilon_0}. \tag{20}$$

Equations (6), (9), (11), (16), and (17) thus determine a new class of steady-state, axisymmetric solutions of the Einstein equations. In the special case $\delta=0$, this solution becomes the Kerr solution. In the other special case $\delta=-2$, $\epsilon_0=0, b=-a$, this solution becomes the solution derived in Refs. 1, 2, and 4 [inversion (5) is used in making this transformation]. By combining the steady-state, axisymmetric solution derived here with the Bonnor theorem, we can derive a corresponding magnetostatic solution, making use of the equations used in Ref. 5 (among other places).

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