Analytic expressions for the contributions of certain five-loop diagrams to the anomalous magnetic moment of the muon

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Analytic expressions are derived for the contributions of two classes of five-loop diagrams, consisting of 6 and 105 diagrams, to the anomalous magnetic moment of the muon.

A standard problem in quantum electrodynamics (QED) is calculating perturbation-theory corrections to the anomalous magnetic moment of the muon, $(g-2)_\mu$, with the further goal of making a precise comparison of theoretical results and experimental data. In recent years, thanks in many ways to the monumental work by Kinoshita, a numerical expression has been constructed for the coefficients of the perturbation-theory series for $(g-2)_\mu$ in the four-loop approximation of QED.[1] It follows from the results found by Kinoshita *et al.*[1] that, in the four-loop approximation, the numerical form of the expression for the difference between the contributions to the anomalous magnetic moments of the muon and the electron is

$$\frac{1}{2}(g_\mu - g_e) = 1,0942596(\alpha/\pi)^2 + 22,8671(33)(\alpha/\pi)^3$$

$$+ 126,92(41)(\alpha/\pi)^4 + O(m_e/m_\mu). \qquad (1)$$

The four-loop coefficient in (1), which incorporates terms which are large in magnitude, on the order of $\ln^i(m_\mu/m_e)$, where $i = 1,2,3$, is itself large. It is thus time to take up the problem of evaluating the various contributions to the five-loop term r_5 in series (1). The first step in this direction was taken by Kinoshita *et al.*,[1] who derived the estimate

$$r_5 = 570(140)(\alpha/\pi)^5. \tag{2}$$

This estimate is based for the most part on the calculations in Ref. 1 of certain five-loop diagrams which contain subdiagrams corresponding to a scattering of light by light. Diagrams of this sort first arise in the expression for $(g-2)_\mu$ at the three-loop level; they are predominant in the corresponding corrections.[2-4,1] The sum of the diagrams of corresponding structure also makes the predominant model-independent contribution to the four-loop coefficients of the renormalization-group QED β function in various renormalization schemes.[5,6]

In the present letter we derive analytic expressions for the contributions to $(g-2)_\mu$ of the sum of the six five-loop diagrams shown in Fig. 1, with two subdiagrams corresponding to a scattering of light by light. The need to calculate these contributions was pointed out by Kinoshita *et al.*[1]

Since the sum of diagrams containing subdiagrams corresponding to a scattering of light by light is independent of the scheme by which the four-loop contributions to the QED β function are renormalized, we can write a renormalized expression corresponding to this sum for the photon polarization operator $\Pi(-q^2)$ in the scheme of subtraction on the mass shell:

$$\Pi(-q^2/m_e^2) = [a_4 + b_4 \ln(-q^2/m_e^2)](\alpha/\pi)^4. \tag{3}$$

The term a_4 is an unknown constant. The coefficient b_4 is found from the results derived in Ref. 5 for the corresponding contributions to the four-loop coefficients of the QED β function. It is given by

$$b_4 = -\left(\frac{11}{36} - \frac{2}{3}\zeta(3)\right). \tag{4}$$

Now using the methods of Ref. 7, we find the analytic relationship between the corresponding five-loop coefficient of the asymptotic expression for $(g-2)_\mu$ and the coefficients found in (3):

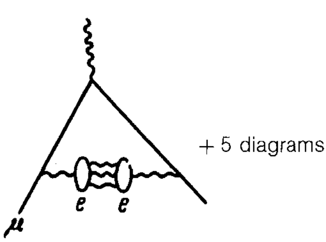

FIG. 1. Class of diagrams formed as the result of an interchange of three photon lines connecting two electron loops.

$$\frac{1}{2}(g-2)_\mu = (-a_4 I_0 - b_4 I_1 - 2b_4 I_0 \ln(m_\mu/m_e) + O(m_e/m_\mu))(\alpha/\pi)^5. \tag{5}$$

The coefficients $I_0 = 1/2$ and $I_1 = -5/4$ were derived in Ref. 7. Substituting (4) into (5), we find

$$\frac{1}{2}(g-2)_\mu = \left(-\frac{a_4}{2} - \left(\frac{55}{144} - \frac{5}{6}\zeta(3)\right)\right.$$
$$\left. + \left(\frac{11}{36} - \frac{2}{3}\zeta(3)\right)\ln(m_\mu/m_e) + O(m_e/m_\mu)\right)(\alpha/\pi)^5. \tag{6}$$

Making use of the tabulated numerical value of the lepton mass ratio,[8] $m_\mu/m_e = 206.768\,262(30)$, we find

$$\frac{1}{2}(g-2)_\mu = \left(-\frac{a_4}{2} - 2,0237 + O(m_e/m_\mu)\right)(\alpha/\pi)^5. \tag{7}$$

In a similar way we find an expression for that part of the five-loop coefficient $(g-2)_\mu$ which corresponds to the class of diagrams shown in Fig. 2. This class consists of 105 diagrams. It contains as a subdiagram a sum of four-loop diagrams for the photon polarization operator Π with one fermion loop. It can be shown that the sum of such diagrams diverges only in a trivial way. Consequently, its contribution to the expression for $\Pi(-q^2)$, renormalized on the mass shell, contains only the relatively unimportant $\ln(-q^2/m_e^2)$ term [see (3)]. We can find the coefficient of this term from the corresponding contribution (which is independent of the renormalization scheme) to the four-loop coefficients of the QED β function:[5,6]

$$b_4 = \frac{23}{128}. \tag{8}$$

Now using (5), we find an expression for the part of the five-loop correction to $(g-2)_\mu$ which is of interest here:

$$\frac{1}{2}(g-2)_\mu = \left(-\frac{a_4}{2} + \frac{115}{512} - \frac{23}{128}\ln(m_\mu/m_e) + O(m_e/m_\mu)\right)(\alpha/\pi)^5. \tag{9}$$

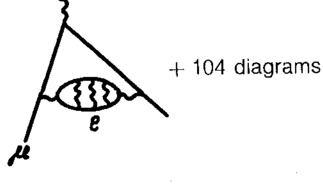

FIG. 2. Class of diagrams formed as the result of an interchange of three photon lines within an electron loop.

Plugging in the numerical value of m_μ/m_e, we find

$$\frac{1}{2}(g-2)_\mu = \left(-\frac{a_4}{2} - 0,7334 + O(m_e/m_\mu)\right)(\alpha/\pi)^5. \qquad (10)$$

The unknown constant term a_4 in expressions (9) and (10) is generally different from the corresponding quantity in expressions (6) and (7). It is unlikely that calculations of the coefficients a_4 in (6) and (9) would substantially alter estimates (7) and (10).

We thus conclude that a numerical calculation of the two classes of diagrams considered here will not alter the estimate in (2) of the overall tenth-order QED contribution to $(g_\mu - g_e)$. This estimate might be refined (first) through numerical calculations of several five-loop diagrams which contain subdiagrams corresponding to a scattering of light by light, and (third) through an analytic calculation of the asymptotic expressions for several five-loop diagrams by the methods of Ref. 7. We hope to return to some of these questions in future work.

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