Reasons why the fluctuations in the multiplicity of groups of cosmic-ray muons do not conform to a Poisson distribution

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A mechanism has been found for fitting a negative binomial distribution to the fluctuations in the multiplicity of the groups of muons.

Distributions of muon groups with respect to multiplicity were first analyzed by Chudakov.[1] He assumed that the fluctuations in the multiplicity of muons generated by nucleons of given energy obey a Poisson distribution. Recent calculations have shown,[2–4] however, that the muon multiplicity fluctuations are not always describable by a Poisson distribution. In general, they can be described by a negative binomial distribution

$$B(n,\bar{n},k)=\binom{n+k-1}{k-1}\left(\frac{\bar{n}/k}{1+\bar{n}/k}\right)^{n}\frac{1}{(1+\bar{n}/k)^{k}}, \tag{1}$$

where $k=\bar{n}^2/(D-\bar{n})$, $\bar{n}$ is the mean value, and D is the variance.

In this letter we wish to point out some factors which would cause a deviation from a Poisson distribution. To do this, we draw on one possible mathematical interpretation of a negative binomial distribution (Ref. 5, for example). We denote by x a random quantity which has a gamma distribution with a density

$$g(x)=\frac{1}{\Gamma(k)}\left(\frac{k}{\bar{n}}\right)^{k}x^{k-1}e^{-\frac{k}{\bar{n}}x},\quad x\geq 0.$$

We denote by n a random quantity which has, at a fixed value of x, a Poisson (P) distribution $P(n)=e^{-x}x^{n}/n!$. A convolution of the gamma and P distributions then gives us a negative binomial distribution:

$$B(n,\bar{n},k)=\int g(x,\bar{n},k)P(x,n)dx. \tag{2}$$

The variance of a negative binomial distribution is given by the following expression, according to the theorem for summing variances: $D=D_g+D_p=\bar{n}^2/k+\bar{n}$.

For events initiated by protons of a given energy, we interpret the negative binomial distribution in (2) in the following way here: $P(x,n)$ is the distribution with

TABLE I.

ν	x	D_x	N[1)]
21 – 40	2.38 ± 0.20	2.04 ± 0.41	50
41 – 60	4.19 ± 0.12	4.47 ± 0.35	326
61 – 80	5.35 ± 0.06	4.99 ± 0.20	1309
81 – 100	6.70 ± 0.05	5.98 ± 0.17	2589
101 – 120	7.78 ± 0.05	7.10 ± 0.20	2415
121 – 140	8.86 ± 0.07	8.47 ± 0.30	1647
141 – 160	9.71 ± 0.11	10.50 ± 0.50	874
161 – 180	10.75 ± 0.16	10.97 ± 0.16	445
181 – 200	11.56 ± 0.24	11.28 ± 1.14	198
201 – 220	12.70 ± 0.37	14.19 ± 2.10	92
221 – 240	12.93 ± 0.60	10.93 ± 2.87	30
241 – 260	14.00 ± 1.18	16.83 ± 7.18	12

[1)] Here N is the number of events.

respect to the number of muons which are produced in events with the same total number (ν) of hadrons (we assume charged pions and kaons) whose energy exceeds the muon detection threshold E_t ($E_\mu \geqslant E_t$); x is the mean number of muons in events with the same number ν; and $g(x,\bar{n},k)$ is the density of events with the mean muon multiplicity x.

To test this assumption, we show in Table I the results of a random generation of 10^4 events initiated by protons with an energy $E = 100$ TeV, with an arrival zenith angle $\theta = 0°$, and with $E_t = 0.22$ TeV (this is the mean threshold for the underground scintillation telescope of the Institute of Nuclear Research, Academy of Sciences of the USSR). The depth of the first proton interaction is the same for all events and corresponds to $z = 50$ g/cm^2. This table compares the mean values (x) and the variances (D_x) of the distributions with respect to the number of muons in events with a number of hadrons in the interval from ν to $\nu + \Delta\nu$. It follows from this comparison that the muon spectra within each interval are described by a Poisson distribution. Figure 1 shows the distribution with respect to the number of events in narrow ν intervals. Plotted along the abscissa here is the mean number of muons in these intervals. We see that the calculated points in this figure can be fitted satisfactorily by a γ distribution with the parameter values $\bar{n}_1 = 7.7$ and $k_1 = 20$. The deviation of the first calculated point from a smooth distribution is attributed to a nonlinearity of the $x(\nu)$ dependence in Table I at small values of ν. We considered the case in which groups of muons are generated in showers initiated by protons at a depth $z = 50$ g/cm^2 in the atmosphere.

Figure 2 shows the z dependence of the parameters. The value of k_1 in this figure is approximately constant. The slight deviation of k_1 from a constant value at small values of z is explained on the basis that the hadron decay probability increases in the atmosphere at high altitudes. As a result, there is a slight deviation from Poisson fluctuations in the narrow ν intervals, since muon production becomes less rare. At $z = 1$ g/cm^2, for example, the probability for this process is 0.13.

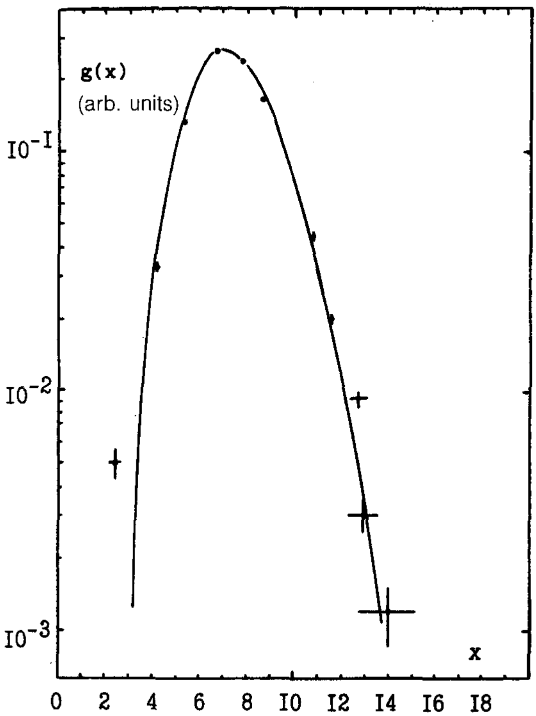

FIG. 1. Approximation of the calculated $N(x)$ dependence in Table I by a gamma distribution $g(x)$.

We turn now to the more general case. Specifically, we carry out an analysis like that above but for the case in which the depth of the first interaction of the primary proton fluctuates in accordance with an exponential law. Figure 3 shows the results of a random generation of 15 000 events initiated by primary protons with $E = 100$ TeV, with $E_t = 0.22$ TeV and $\theta = 0°$. A comparison of the values of $\bar{n}_2$ and D_2 in this figure shows that, when the additional source of fluctuations is "turned on," the result is a non-Poissonian distribution of events with respect to the number of muons in the narrow ν intervals. To interpret this result, we assume, by analogy with (2), that the fluctuations due to the difference between the ranges for the primary interaction are described by a gamma distribution with a ν-independent value $k_2 = \bar{n}_2^2/(D_2 - \bar{n}_2)$. This interpretation is supported by the results shown in Fig. 3. The generalization of (2) for this case is

$$B(n,\bar{n},k) = \int g(y,\bar{n},k_2)dy \int g(x,y,k_1)\frac{e^{-x}x^n}{n!}dx, \tag{3}$$

where $k = f(k_1,k_2)$. This integral converges on a confluent hypergeometric function. It is not a tabulated integral. We accordingly pursue the analysis in the following way. A negative binomial distribution is characterized by the parameters $\bar{n}$ and k. To determine the parameter k in (3), we calculate its variance in terms of the characteristic function

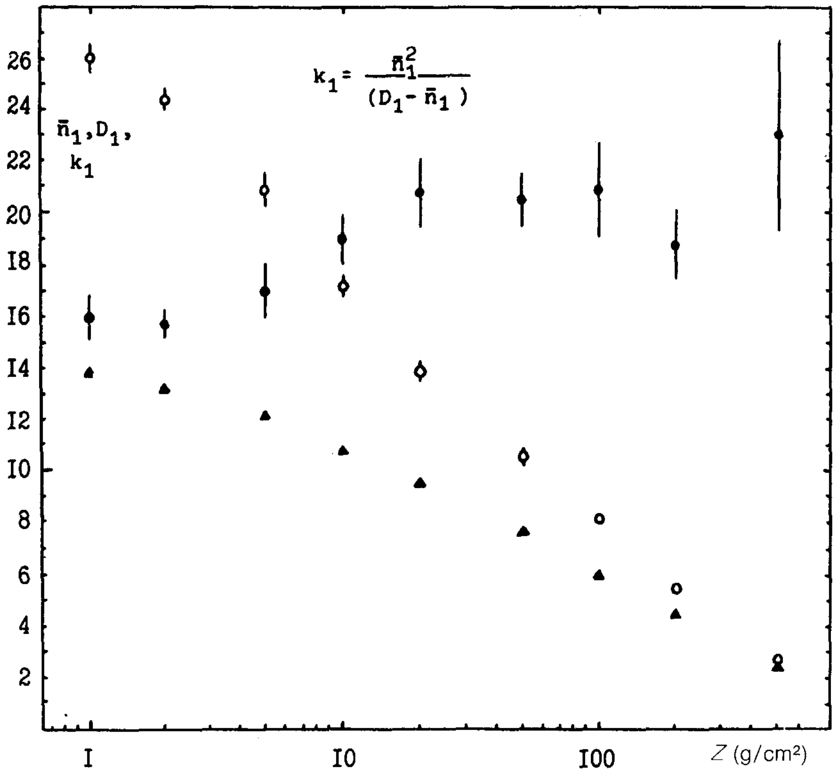

FIG. 2. The parameters $\bar{n}_1(\triangle)$, $D_1(\bigcirc)$, and $k_2(\bullet)$ versus the depth z.

$$\varphi(t) = \sum_{n=0}^{\infty} e^{itn} B(n, \bar{n}, k) = \left(\frac{k_2}{\bar{n}}\right)^{k_2} \frac{k_1^{k_1}}{\Gamma(k_2)} \int y^{k_2-k_1-1} e^{-\frac{k_2}{\bar{n}} y} \frac{dy}{\left(\frac{k_1}{y} + 1 - e^{it}\right)^{k_1}}. \tag{4}$$

The variance is expressed in terms of the derivatives of $\varphi(t)$ at the point $t=0$ (Ref. 6):

$$D = -\varphi''(0) + [\varphi'(0)]^2.$$

Here we have $\varphi'(0) = i\bar{n}$, $\varphi''(0) = -\bar{n}^2(1+k_1)(1+k_2)/k_1k_2 - \bar{n}$, and

$$D = \bar{n}^2 \frac{1 + k_1 + k_2}{k_1 k_2} + \bar{n}. \tag{5}$$

Comparing (5) with the variance of a negative binomial distribution, we find $k = k_1k_2/1 + k_1 + k_2$.

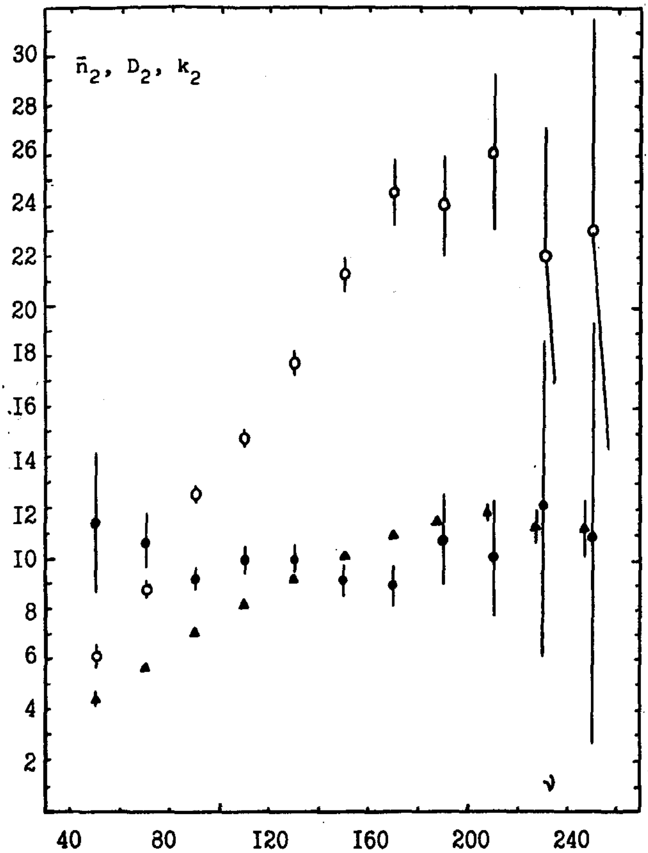

FIG. 3. The parameters $\bar{n}_2(\Delta)$, $D_2(\bigcirc)$, and $k_2(\bullet)$ versus ν in the case of fluctuations in z.

We have thus derived an expression for the parameter k of a negative binomial distribution in terms of the parameters k_1 and k_2.

We now find the square of the relative fluctuations for a negative binomial distribution in terms of the parameters k_1, k_2, and $\bar{n}$. From the definition of the variance of a negative binomial distribution and from (5), we have

$$\delta^2 = \frac{D}{\bar{n}^2} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_1 k_2} + \frac{1}{\bar{n}}. \tag{6}$$

For a quantitative estimate of δ^2, we make use of the results shown in Figs. 2 and 3. From the calculated value of the mean multiplicity, $\bar{n} \simeq 8.0$, and from the values $k_1 \simeq 20$ and $k_2 \simeq 9.5$, we find $\delta^2 \simeq 0.28$. In this case, the value of δ^2 is dominated by the $1/\bar{n}$ term.

Three processes are thus influencing the fluctuations in the multiplicity of the muon groups: (1) fluctuations in the shower nucleation depth, (2) fluctuations in the total number of hadrons by means of which muons form, and (3) fluctuations in the decays of hadrons. The latter source leads to fluctuations in the muon multiplicity which are described approximately by a Poisson distribution, as was shown above. The presence of the first two sources of fluctuations, however, causes a deviation from a Poisson distribution and leads to a more accurate fit of the muon multiplicity spectrum by a negative binomial distribution.

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