Distinctive features of the energy distribution of particles interacting with a strong long-wave turbulence

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Energy distributions of nonthermal charged particles are derived for the case in which nonperturbative effects stem from the nonzero change in the energy of a particle over the correlation length of the random electromagnetic field in the plasma. The steady-state energy distribution of the particles has a universal logarithmic singularity at small values of the momentum transfer.

The kinetics of charged particles in random electromagnetic fields in various physical systems has been studied in detail in the Fokker–Planck approximation.[1,2] In general, the condition for the applicability of this approximation reduces to the requirement that the change in the momentum and/or the energy of the particle over the correlation length L_c or over the correlation time τ_c of the random field be small. The Fokker–Planck approximation is thus valid only for particles with a fairly high energy or for systems in which the fields have a short correlation length. The Fokker–Planck approximation is not adequate for solving many problems, e.g., the shaping of the energy spectra of low-energy cosmic rays ($E \leqslant 1$ GeV) in the local galaxy. In this letter we wish to discuss some particular features of the energy distributions of charged particles which stem from the large change in the energy of the particles over the correlation length in a system with a strong long-wave MHD turbulence.

We consider a plasma system with a well-developed, statistically homogeneous, isotropic MHD turbulence. The electric fields induced by the turbulent motions of an ideally conducting medium with a frozen-in magnetic field lead to a statistical change in the energy of superthermal charged particles. Fluctuations in the magnetic field with length scales smaller than or on the order of the gyroradius of the particle cause the momenta of the particles to become effectively isotropic (in the absence of Coulomb collisions). These fluctuations also determine the transport mean free path λ of the particles. We will say that fluctuations with length scales $L_c \gg \lambda$ are "long-wave" fluctuations. The distribution function $G(\vec{r},\eta,t)$ of the superthermal particles, averaged over the ensemble of turbulent fluctuations of all length scales, satisfies an equation derived in Ref. 3:

$$\frac{\partial G}{\partial t} - \int_{-\infty}^{\infty} d\eta' \chi_{\alpha\beta}(\eta - \eta') \frac{\partial^2 G(\vec{r}, \eta', t)}{\partial r_\alpha \partial r_\beta} - \left(\frac{\partial^2}{\partial \eta^2} + 3\frac{\partial}{\partial \eta}\right) \int_{-\infty}^{\infty} D(\eta - \eta') G(r, \eta', t) d\eta' = Q\delta(\eta). \quad (1)$$

We have replaced the particle momentum p by the variable $\eta = \ln(p/p_0)$; Q is the rate at which monoenergetic particles with a momentum p_0 are generated by the steady-state source. It is convenient to solve Eq. (1) in the Fourier representation in the variable η. We denote the transform of this variable by s. The Fourier transforms of the kernel of integral equation (1) are expressed in terms of the correlation functions of the turbulence.[3]

We consider the extremely general case of a compressible system with a broad spectrum of velocity fluctuations:

$$W(k) = \frac{< u^2 >}{4} C(\nu) \frac{k_0^{\nu-1}}{(k^2 + k_0^2)^{\nu/2+1}}, \qquad k \leq k_{max} . \tag{2}$$

We assume that the system has a Lorentzian frequency dependence characterized by a dispersion relation $\omega_0 = uk$, and we assume that it has a "resonance width" $\gamma/2 = uk_0(k/k_0)^{(3-\nu)/2}$, where $u \equiv \langle u^2 \rangle^{1/2}$, and $C(\nu) \approx 4\Gamma(\nu/2+1)/3\pi^{3/2}\Gamma(\nu/2 - 1/2)$ is a normalization constant. In this case, following (3), and allowing for the nonzero change in the energy of a particle over the field correlation length, we find a system of transcendental equations for the Fourier transforms of the kernels:

$$D_1(s) = \epsilon + C(\nu) \int_0^b \frac{x^2 dx}{(x^2+1)^{\nu/2+1}} \left\{ \frac{\psi}{\psi^2 + \omega_0^2} - \frac{2}{3} D_1 x^2 \left(\frac{\lambda}{6} + 1 \right) \right.$$
$$\left. \times \frac{\psi^2 - \omega_0^2}{(\psi^2 + \omega_0^2)^2} + \frac{4}{9} \lambda D_1^2 x^4 \frac{\psi^3 - 3\psi\omega_0^2}{(\psi^2 + \omega_0^2)^3} \right\}, \tag{3}$$

$$D_2(s) = \frac{C(\nu)}{9} \int_0^b \frac{x^4 dx}{(x^2+1)^{\nu/2+1}} \frac{\psi}{\psi^2 + \omega_0^2}. \tag{4}$$

The dimensionless variables in (3) and (4) have been introduced in the following way: $\chi_{\alpha\beta}(s) \equiv uk_0^{-1} D_1(s) \delta_{\alpha\beta}$, $D(s) \equiv uk_0 D_2(s)$, $b = k_{\max}/k_0$, $x = k/k_0$, $\epsilon = \kappa k_0/u$. Here $\kappa(\kappa \propto \lambda)$ is a "seed" diffusion coefficient which arises from the scattering of particles by small-scale fluctuations of the magnetic field, and

$$\psi(x, \lambda) = D_1(s) x^2 + \lambda D_2(s) + \frac{\gamma(x)}{2}, \quad \lambda \equiv s(s + 3i). \tag{5}$$

Kinetic equation (1), with the kernels found from (3) and (4), describes the energy distributions which are shaped through a stochastic change in the energy of the particles (which are initially monoenergetic) by the long-wave fluctuations of the electric field. The derivation of Eq. (1) rests on the assumption that the velocities of the superthermal particles are greater than the turbulent velocities u of the medium. The parameter ϵ can be arbitrary in this case.

If $\epsilon > 1$, i.e., if the particle transport due to scattering by small-scale fluctuations of the magnetic field is predominant, then the solutions of Eqs. (3) and (4) can be approximated by $D_1(s) \approx \epsilon$ and $D_2(s) \approx (9\epsilon)^{-1}$. We thus have $\chi_{\alpha\beta}(\eta - \eta') \approx \kappa\delta(\eta - \eta')\delta_{\alpha\beta}$ and $D(\eta - \eta') \approx u^2(9\kappa)^{-1}\delta(\eta - \eta')$. Equation (1) then reduces to a Fokker–Planck equation.

If $\epsilon \ll 1$, the particle transport is determined by long-wave fluctuations of the velocity of the medium. In this case, the renormalization of the kernels of Eq. (1) is important. The s dependence of D_1 and D_2 is not trivial. If $s \leqslant 3$, the transforms of the kernels are only weak functions of s: $D_1(s) \approx D_1(0)$ and $D_2(s) \approx D_2(0)$. The behavior then depends strongly on the width of the fluctuation spectrum, which is characterized by the parameter $b (b > 1)$. If $\epsilon b^{(\nu+1)/2} < 1$, then with $s \leqslant a(\nu)$ [where $a(\nu) = \sqrt{9(3-\nu)/C(\nu)} \sim 10$] the s dependence of the kernels is weak. If $s \gg a(\nu)$, we have the following asymptotic expressions for the Fourier transforms of the kernels:

$$D_1(s) \rightarrow \epsilon/2, \tag{6}$$

$$D_2(s) \rightarrow b^{(3-\nu)/2}(a(\nu)\sqrt{\lambda})^{-1}. \tag{7}$$

In the case of a wide fluctuation spectrum $\epsilon b^{(\nu+1)/2} \gg 1$, the asymptotic expressions in (6) and (7) hold for $s \gg a(\nu)\epsilon b^{(\nu-1)/2}$. The behavior of the functions $D_1(s)$ and $D_2(s)$ at $s \geqslant 3$, but before the asymptotic behavior sets in, depends on the particular shape of the "resonance width" $\gamma(k)$ and also on the parameters ϵ and b. In particular, the behavior $D_2(s)$ can be nonmonotonic in this region, with several maxima. The limiting value in (6) is typically smaller than the seed diffusion coefficient ϵ. This is a general property, which stems from the compressibility of the system.[4]

We can use analytic expressions (6) and (7) to analyze the energy distributions of the charged particles at momenta close to the injection momentum. For simplicity, and without detracting seriously from the generality of the results, we can replace the spatial-diffusion operator in (1) by its eigenvalue. The Fourier transform of the steady-state particle distribution function (a Green's function in this case) is then

$$G(s) = \frac{Q(uk_0)^{-1}}{\xi D_1(s) + \lambda D_2(s)}, \tag{8}$$

where $\xi \approx (k_0 R)^{-2}$, and R is the size of the region occupied by the turbulent fluctuations. Using (6)–(8), we find an asymptotic expression for the distribution function of the particles for momenta close to the injection momentum p_0:

$$G(p, p_0) \propto -Q(uk_0)^{-1} b^{(\nu-3)/2} a(\nu) \ln|\ln(p/p_0)|, \quad p \rightarrow p_0. \tag{9}$$

Distribution (9) has an integrable logarithmic singularity near p_0, because of the possible finite change in the energy of the particle over the correlation length. At moment $p \gg p_0$ the particle distribution in (8) is determined by the behavior of $D_1(s)$ and $D_2(s)$ as $s \rightarrow 0$:

$$G(p, p_0) \propto Q(uk_0 D_2(0))^{-1}(p/p_0)^{\mu}, \quad \mu = -1.5 - (2.25 + \xi D_1(0) D_2^{-1}(0))^{0.5}. \tag{10}$$

Equations (10) describe a Fokker–Planck spectrum for all $p \gg p_0$. As $p \to p_0$, the Fokker–Planck distribution function differs from (9) in having a finite limit and a weak dependence on the width of the fluctuation spectrum, b.

We note in conclusion that the results described above can be used to calculate the distribution of photons produced by Thomson scattering in an optically dense medium with macroscopic turbulent motions. In this case the seed diffusion coefficient κ is determined by the electron density (Ref. 5, for example).

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