Master equation for conformal field theories with auxiliary symmetries

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The general form of the master equation is found for conformal field theories with Z_3 symmetry under the condition that the fields which generate the W_3 algebra belong to a universal envelope algebra of an arbitrary Kac–Moody algebra.

A classification of conformal field theories lays a foundation for the construction of an effective Lagrangian string theory. It is also a necessary step toward a fuller understanding of the entire set of critical phenomena of 2D field-theory models. A huge number of conformal field theories have now been derived. Most are consistent with a WZNW interpretation. However, an even greater number of conformal theories have not yet been observed, or no explicit field-theory realization is known for them. In this connection, it would be useful to study equations for the structure constants of a quantum field theory constructed in the form of a specific ansatz, say, a polynomial ansatz in terms of current algebra—equations which involve conformal invariance.

A systematic search for new conformal field theories is based on analysis of the affine-Virasoro master equation (AVME), which was found first by Halpern and Kiritsis[1] and, independently, by Morozov *et al.*[2] (see also Refs. 3–5, where various solutions of the AVME were found). In this letter we construct a master equation for conformal theories which have an auxiliary symmetry (for definiteness, Z_3 symmetry).

Let us construct a generalization of the master equation to the case of the Zamolodchikov W_3 algebra, which we denote by AW_3ME. For this purpose we introduce, along with $T(z)$, a chiral field $Q(z)$ of spin 3:

$$T(z)Q(w) = \frac{3}{(z-w)^2}Q(w) + \frac{1}{z-w}\partial Q(w) + \text{reg.t.} \tag{1}$$

We construct this chiral field as a trilinear form on the G^k current algebra:

$$Q(z) = q_{abc} : J^a J^b J^c : (z), \tag{2}$$

where q_{abc} is the completely symmetric, traceless G tensor. The operator expansions for the fields $Q(z)$ and $Q(w)$ are

$$\begin{aligned} Q(z)Q(w) = & \frac{c}{3(z-w)^6} + \frac{2T(w)}{(z-w)^4} + \frac{\partial T(w)}{(z-w)^3} \\ & + \frac{1}{(z-w)^2}\left\{2b^2\Lambda(w) + \frac{3}{10}\partial^2 T(w)\right\} \\ & + \frac{1}{z-w}\left\{b^2\partial\Lambda(w) + \frac{1}{15}\partial^3 T(w)\right\} + \text{reg.t.}, \end{aligned} \tag{3}$$

where

$$\Lambda(w) = (TT)(w) - \frac{3}{10}\partial^2 T(w) \quad \text{and} \quad b^2 = \frac{16}{22+5c} \, .$$

To derive a master equation which selects from fields of the type in (2) the subset which satisfies, along with an energy-momentum tensor

$$T(z) = t_{ab} : J^a J^b : (z),$$

the W_3 algebra, we should substitute ansatz (2) into operator expansions (1) and (3). We should then make repeated use of Wick's theorem for the reduction of the operator structures $\langle AB\rangle_k(z)$ which arise on the right side of the operator expansion for composite operators $A(z)$ and $B(z)$.

We start the derivation of the AW$_3$ME system with the $T{\circ}Q \to Q$ channel. We consider the $(z-w)^{-1}$ level. At this level we have operator structures of the following types: $:J^aJ^bJ^cJ^d:(z)$, $(\partial J^a :J^bJ^c:)(z)$, $(J^a\partial^2 J^b)(z)$, $:\partial J^a \partial J^b:(z)$ and $\partial^3 J^a(z)$. The disappearance of the coefficient of the tetralinear form leads to the equation

$$S_{stuv} t_{ab} q_{lmn} f_s^{al} \delta_t^b \delta_u^m \delta_v^n = 0. \tag{4}$$

The requirement that the term $\simeq :\partial J^a \partial J^b:(z)$ vanish leads to the equation

$$t_{ab} q_{lmn} f_x^{al} f_y^{xm} f_s^{yn} \delta_t^b = 0. \tag{5}$$

From the condition that the coefficient of $(\partial J^a :J^bJ^c:)(z)$ be same as q_{abc} we find

$$\begin{aligned} S_{st} t_{ab} q_{lmn} \{ & -2 f_x^{al} f_s^{xm} \delta_u^b \delta_t^n + f_x^{al} f_u^{xb} \delta_s^m \delta_t^n \\ & -2 f_s^{al} f_u^{bm} \delta_t^n + 2k\eta^{al} \delta_u^b \delta_s^m \delta_t^n \} = q_{stu}. \end{aligned} \tag{6}$$

Let us look at an operator structure of the type $(J^a\partial^2 J^b)(z)$. The vanishing of the symmetric part $\simeq S_{ab}(J^a\partial^2 J^b)(z)$ leads to an equation which is a consequence of Eqs. (4)–(6), while the vanishing of the antisymmetric part $\simeq A_{ab}(J^a\partial^2 J^b)(z)$ leads to an equation of the type

$$A_{st}t_{ab}q_{lm}\{\alpha f^{al}_x f^{xm}_y f^{yn}_s \delta^b_t + \beta f^{al}_x f^{xb}_y f^{ym}_t \delta^n_s$$
$$+\gamma f^{al}_x f^{xm}_s f^{bn}_t + \delta k f^{al}_t \ \eta^{bm}\delta^n_s\} = 0, \qquad (7)$$

where $(\alpha,\beta,\gamma,\delta) = (-1,5,-1,6)$.

We thus see, that, together, Eqs. (4)–(6) and (7) constitute the complete system which we need in order to satisfy condition (1), under which the field $Q(z)$ is primary.

We turn now to the $Q\circ Q\rightarrow T$ channel. Substituting ansatz (2) into Eq. (3), and going through an analysis like that above, we find that the complete system of equations for q_{abc} which we need in order to satisfy W_3 algebra (3) consists of the following four equations:

$$S_{stuv}\{9q_{abc}q_{lmn}[2f^{al}_x f^{xb}_s \delta^c_t \delta^m_u \delta^n_v - 2f^{al}_s f^{bm}_t \delta^c_u \delta^n_v$$
$$+k\eta^{al}\delta^b_s\delta^c_t\delta^m_u\delta^n_v] - 2b^2 t_{st}t_{uv}\} = 0 \qquad (8a)$$

$$S_{stu}\ \{6q_{abc}q_{lmn}[f^{al}_x f^{xb}_y f^{yc}_s \delta^m_t \delta^n_u - f^{al}_x f^{xb}_y f^{yc}_u \delta^m_s \delta^n_t$$
$$+3f^{al}_x f^{xb}_y f^{ym}_s \delta^c_u \delta^n_t - 3f^{al}_x f^{xb}_s f^{cm}_t \delta^n_u$$
$$+3f^{al}_x f^{xb}_u f^{cm}_s \delta^n_t + 6kf^{al}_s \eta^{bm}\delta^c_u \delta_t{}^n]$$
$$-8b^2 t_{ab}t_{lm} f^{al}_s \delta^b_u \delta^m_t\} = 0 \qquad (8b)$$

$$A_{st}\{9q_{abc}q_{lmn}[2f^{al}_x f^{xb}_y f^{ym}_z f^{zc}_s \delta^n_t - f^{al}_x f^{xb}_y f^{ym}_s f^{cn}_t$$
$$+2kf^{al}_x f^{xb}_s \eta^{cm}\delta^n_t] - 4b^2 t_{ab}t_{lm} f^{al}_x f^{xb}_s \delta^m_t\} = 0 \qquad (8c)$$

$$S_{st}\{9q_{abc}q_{lmn}[2f^{al}_x f^{xb}_y f^{ym}_z f^{zc}_s \delta^n_t - 2f^{al}_x f^{xb}_y f^{ym}_s f^{cn}_t$$
$$+f^{al}_x f^{xb}_s f^{cm}_y f^{yn}_t + 4kf^{al}_x f^{xb}_s \eta^{cm}\delta^n_t + kf^{al}_x f^{xbm}\delta^c_s \delta^n_t$$
$$-2kf^{al}_s f^{bm}_t \eta^{cn} + 2k^2\eta^{al}\eta^{bm}\delta^c_s\delta^n_t]\} = 2t_{st}. \qquad (8d)$$

Equations (4)–(8), along with the AVME,

$$kt_{ab}t_{cd}(f^{ac}_s f^{sbd} + 2k\eta^{ac}\eta^{bd}) = c/2$$

and

$$S_{mn}t_{ab}t_{cd}(2f^{ac}_{s}f^{sb}_{m}\delta^{d}_{n} - f^{ac}_{m}f^{bd}_{n} + 2k\eta^{ac}\delta^{b}_{m}\delta^{d}_{n}) = t_{mn} \,,$$

constitute the complete AW_3ME system.

In a corresponding way, we can derive AW_3ME equations for an expansion of ansatz (2) including $(J^a\partial J^b)(z)$ and $\partial^2 J^a(z)$ terms.

Analysis of the AW_3ME system leads to the following conclusions regarding the structure of conformal field theories with Z_3 symmetry.

1. *Any* affine model of a W_3 algebra which has a central charge c is generated by an affine-Virasoro model (a W_2 algebra), which has a central charge $\tilde{c} = (c+2)/4$, in the sense that the fields $T(z)$ and $Q(z)$ which form the W_3 algebra are differential polynomials of the energy-momentum tensor $T(z)$ and the "Romans field"[7] $H_R(z)$:

$$T(z) = \tilde{T}(z) + T_R(z)\,,$$

$$Q(z) = b\sqrt{2}((H_R + \frac{1}{2}a\partial)(\tilde{T} - \frac{1}{3}T_R))(z), \tag{9}$$

where $T_R(z) = \frac{1}{2}(H_R H_R)(z) + a\partial H_R(z)$.

The classification of affine W_3 algebras is thus completely similar to the classification of affine-Virasoro models. Previously this similarity had been established only for the special case of minimal W_3 models ($c<2$).

A corollary of the isomorphism of the classifications of W_2 and W_3 affine models is the assertion that the Romans hypothesis[7] of the existence of "magic" quantum W_3 algebras—corresponding to four magic Jordan algebras J^A, $A\in\{R,C,H,O\}$, of dimensionality $n = 5$, 8, 14, 26, respectively—is incorrect. Consequently, a Jordan nature is a specific property of only classical W_3 algebras. There are accordingly some exceptional classical W_3 algebras, which cannot be continued to the quantum level. This is an important point for an analysis of W gravities.

3. The deformable affine-Virasoro CFT models with a modulus space $G_{\tilde{c}}(R^r)$, which were found in Ref. 8, can immersed in affine W models with a central charge $c = 4\tilde{c} - 2$. This result confirms Morozov's hypothesis[9] that there exist multiparameter deformations of W algebras.

4. With $c = 2$, the Romans construction[7] has a hidden symmetry, which leads to an expansion of the deformation space, $G_1(R^r) \to G_2(R^r)$, and to the possibility of an Abelization of W_3 constructions.

5. The renormalization-group (RG) flux in the space of affine-Virasoro models[10] allows a natural continuation to W_3 models within the framework of the Romans construction. Specifically, the RG connection of two affine-Virasoro models with central charges $\tilde{c}_1$ and $\tilde{c}_2$ implies an RG connection of the W_3 models with central charges c_1 and c_2 ($c_i = 4\tilde{c}_i - 2$) which they generate.

6. A generalization of AW_3ME to the case of higher W algebras makes it possible to find some direct analogs of the assertions made above for all algebras W_{g_L}, where g_L is a Lie algebra of rank L with simple constraints (of the ADE type). In particular,

with $c = L$, the hidden symmetry of the generalized Romans construction leads to an expansion of the deformation space:

$$G_1(R^r) \to G_L(R^r).$$

We note in conclusion that the presence of conformal theories with auxiliary symmetries which have multiparameter marginal deformations is a fundamental factor in the formulation of a dynamic principle in the space of models which includes (as points) conformal theories and, along with them, integrable models which can be interpolated between them. This circumstance is a constituent part of the program for constructing a *complete* string theory (i.e., off the mass shell). The presence of the sporadic deformations mentioned above implies, from the standpoint of an effective dynamic theory in model space, the existence of some specific "zero modes" which have all kinds of expanded symmetries.

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