# Nonperturbative boson string and Diff$S^1/SL(2,R)$

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The space of complex structures on the phase space of an open boson string, which plays a leading role in one of the nonperturbative approaches to string quantization, is isomorphic with respect to the Kähler manifold Diff$S^1/SL(2,R)$, not with respect to Diff$S^1/S^1$, as was asserted in some recent papers by Bowick and Rajeev.

Bowick and Rajeev[1] have proposed a new geometric and nonperturbative approach to the boson string theory. In their approach, a leading role is played by the space $\mathscr{M}$ of all complex structures on the phase space of the open boson string, $\Omega R^{d-1,1}$. In this letter we show that a natural complex structure $J$ on $\Omega R^{d-1,1}$ is invariant not only under rotations $S^1$ but also under the less obvious symmetry group $SL(2,R)\subset$ Diff$S^1$. We draw an important conclusion from this result: The space $\mathscr{M}$ is isomorphic with respect to the uniform Kähler manifold Diff$S^1SL(2,R)$, not with respect to the manifold Diff$S^1/S^1$, as was asserted in Refs. 1 and some other papers.[2-4] At the end of this letter we will discuss the physical meaning of this conclusion.

We denote by $\mathscr{L}R^{d-1,1}$ the space of maps of a circle into Minkowski $R^{d-1,1}$, and we denote by $\Omega R^{d-1,1}\equiv\Omega R^{d-1,1}/R^{d-1,1}$ the space of loops with the point which we just mentioned. It was shown in Ref. 1 that $\Omega R^{d-1,1}$ can be identified with the phase space of an open boson string or with the configuration space of a closed boson string.

Corresponding to the vector fields $ie^{ik\sigma}(d/d\sigma)$, $k\in Z$, on $S^1$ are the vector fields

$$L_k = i\int d\sigma e^{-ik\sigma}\frac{dx^\mu(\sigma)}{d\sigma}\frac{\delta}{\delta x^\mu(\sigma)} \tag{1}$$

on $\mathscr{L}R^{d-1,1}$, which realize a representation of the Lie algebra Diff$S^1$:

$$[L_m,\ L_n] = (m-n)L_{m+n}. \tag{2}$$

We denote by $J$ some Lorentz-invariant complex structure on $\mathscr{L}R^{d-1,1}$. This structure is conveniently represented by an integral operator:

$$J(W) = \int d\sigma\{\int d\sigma' J^\mu_\nu(\sigma,\ \sigma')W^\nu(\sigma')\}\frac{\delta}{\delta x^\mu(\sigma)}, \tag{3}$$

where $W=\int d\sigma W^\nu(\sigma)[\delta/\delta x_\nu(\sigma)]$ is an arbitrary vector field on $\mathscr{L}R^{d-1,1}$. Since the complex structure $J$ is Lorentz-invariant, its integral kernel should be of the form $J^\mu_\nu(\sigma,\sigma') = \delta^\mu_\nu J(\sigma,\sigma')$ for some function $J(\sigma,\sigma')$ on $S^1\times S^1$ [the integral in (3) is to be understood in the principal-value sense]. The condition of a complex nature,

  

$J^2 = -id$, then takes the form

$$\int d\sigma' J(\sigma, \sigma') J(\sigma', \sigma'') = -\delta(\sigma - \sigma''). \tag{4}$$

Our next task is to find those complex structures on $\mathscr{L}R^{d-1,1}$ which are invariant under the vector fields $L_0$ and $L_k$ for some $k \in Z \setminus 0$, i.e., which satisfy the equations

$$\mathcal{L}_{L_0} J = 0, \qquad \mathcal{L}_{L_k} J = 0, \tag{5}$$

where $\mathscr{L}_W$ denotes the Lie derivative along the vector field $W$. Straightforward calculations show that Eqs. (5) are equivalent to the following differential equation for the kernel $J(\sigma,\sigma') = J(\sigma - \sigma')$ of complex structure (3):

$$[e^{-ik(\sigma-\sigma')} - 1]\frac{d}{d\sigma} J_k(\sigma - \sigma') = ikJ_k(\sigma - \sigma'), \quad k \in Z\setminus 0. \tag{6}$$

It follows that we have

$$J_k(\sigma - \sigma') = A\{1 - e^{-ik(\sigma-\sigma')}\}^{-1} = \frac{A}{2}[1 + i\cot(\frac{1}{2}k\sigma)] \tag{7}$$

for a constant $A$. It is not difficult to verify that among functions (7) only the functions $J_k(\sigma - \sigma')$ with $k = \pm 1$ can be normalized in such a way that they satisfy Eq. (4). Consequently, there exists a unique complex structure $J_{+1}$ on $\mathscr{L}R^{d-1,1}$, with the kernel

$$J_{+1}(\sigma - \sigma') = i[1 + i\cot\frac{1}{2}(\sigma - \sigma')] = i\sum_{m\geq 0} e^{im\sigma} - i\sum_{m<0} e^{im\sigma}, \tag{8}$$

which is invariant under $L_0$ and $L_1$. There also exists a unique complex structure $J_1$ with the kernel

$$J_{-1}(\sigma - \sigma') = ie^{i(\sigma-\sigma')}[1 + i\cot\frac{1}{2}(\sigma - \sigma')] = i\sum_{m\geq 1} e^{im\sigma} - i\sum_{m<1} e^{im\sigma}, \tag{9}$$

which is invariant under $L_0$ and $L_{-1}$.

If $W^\mu(\sigma) = \Sigma_{n\in Z} W^\mu_n e^{in\sigma}$ are the components of some vector on $\mathscr{L}R^{d-1,1}$, the effect of complex structures $J_{-1}$ and $J_{+1}$ can be written explicitly as follows:

$$(J_{+1}(W))^\mu = +iW^\mu_0 + i\sum_{n\neq 0} \mathrm{sgn}(n) W^\mu_n e^{in\sigma}, \tag{10}$$

$$(J_{-1}(W))^\mu = -iW^\mu_0 + i\sum_{n\neq 0} \mathrm{sgn}(n) W^\mu_n e^{in\sigma}. \tag{11}$$

It follows from (10) and (11) that the $(L_0,L_1)$-invariant complex structure $J_{+1}$ and the $(L_0,L_{-1})$-invariant complex structure $J_{-1}$ differ only in the effect which they have on the zero mode $W^\mu_0$. Consequently, on the factor space

$\Omega R^{d-1,1} = \mathscr{L} R^{d-1,1}/R^{d-1,1}$ they induce the same complex structure $J$, which is exactly the same as the unique complex structure[1,5] on $\Omega R^{d-1,1}$. Another conclusion is that $J$ is invariant under not only the rotation group $S^1$ but also the group $SL(2,R)$, which is generated by the vector fields $L_{-1}$, $L_0$, and $L_{+1}$. We furthermore conclude that this is the only complex structure on $\Omega R^{d-1,1}$ which has this property. Consequently, the manifold of all complex structures on $\Omega R^{d-1,1}$ is isomorphic with respect to the uniform Kähler manifold Diff$S^1/SL(2,R)$, not with respect to Diff$S^1/S^1$, as was asserted in Refs. 1 and in some subsequent papers.[2-4] It can be shown that this conclusion does not alter the interpretation of Ref. 1 regarding a critical dimension of 26 as the condition for the existence of holomorphic and horizontal sections of the vacuum stratification $B \oplus \Gamma$ above Diff$S^1/SL(2,R)$.

Our conclusion is completely consistent with the results found by Pilch and Warner,[3] who used some mathematical facilities different from those in Refs. 1 to study the vacuum stratification $B \oplus \Gamma$ above Diff$S^1/S^1$. They showed that the vacuum states of a boson string are described by $SL(2,R)$-invariant sections of stratification $B \oplus \Gamma$ above Diff$S^1/S^1$. The mutual consistency of this result and of our own conclusion can be explained in the following way: The $SL(2,R)$-invariant holomorphic and horizontal sections of the vacuum stratification above Diff$S^1/S^1$ are exactly the elevations [with respect to the natural projection Diff$S^1/S^1 \to$ Diff$S^1/SL(2,R)$] of the holomorphic and horizontal sections of the vacuum stratification above Diff$S^1/SL(2,R)$.

Support was found in Refs. 4 for the hypothesis that a nonperturbative string amplitude can be written as an integral over $\mathscr{M} =$ Diff$S^1/SL(2,R)$ with a measure determined by the Kähler metric on $\mathscr{M}$. On the other hand, Diff$S^1/SL(2,R)$ is a dense complex submanifold[4] of the universal Teichmüller space, which contains all the Teichmüller spaces $T_g$, $g \geqslant 1$, which correspond to Riemannian surfaces of kind $g$. For this reason, a nonperturbative approach to the quantization of boson strings based on Diff$S^1/SL(2,R)$ must be intimately related to the standard approach, which involves a summation over the kinds $g$.


[1]M. J. Bowick and S. G. Rajeev, Phys. Rev. Lett. **58**, 535 (1987); Nucl. Phys. B **293**, 348 (1987).
[2]D. Harari, D. K. Hong, P. Ramond, and V. Rodgers, Nucl. Phys. B **294**, 556 (1987); Z. Y. Zhao, K. Wu, and T. Saito, Phys. Lett. B **199**, 37 (1987); J. Mickelsson, Commun. Math. Phys. **112**, 653 (1987); E. Witten, Commun. Math. Phys. **144**, 1 (1988); M. A. Awada and A. H. Chamseddine, Phys. Lett. B **206**, 437 (1988); Y. Yu, Phys. Lett. B **216**, 75 (1989).
[3]K. Pilch and N. P. Warner, Class. Quantum Grav. **4**, 1183 (1987).
[4]S. Nag and A. Verjovski, Commun. Math. Phys. **130**, 123 (1990); D. K. Honge and S. G. Rajeev, Commun. Math. Phys. **135**, 401 (1991).
[5]A. N. Pressley, J. London Math. Soc. **26**, 557 (1982); A. Pressley and G. Segal, *Loop Groups*, Oxford U. Pr., London, 1988.


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