The SU(3) Black Hole

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The Einstein–Yang–Mills equations solution which is a SU(3) black hole was built in this article.

The Einstein's gravity theory considers a class of asymptotic flat gravitational fields, which are created by the mass (the Schwarzschild's solution), by the mass and charge (the Reissner–Nordström's solution), and by the rotated mass (the Kerr's solution). A similar solution, which describes the black holes with the gauge SU(2) fields, was recently obtained.[1]

In this article the solution corresponding to the black holes with the gauge SU(3) field is given.

Statement of the problem

The metric is chosen in the following spherically symmetric form:

$$ds^2 = \frac{\sigma^2 u}{r^2} dt^2 - \frac{r^2}{u} dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \qquad (2.1)$$

where the functions σ and u depend on r.

We introduce the Cartesian coordinates

$$x^1 = r\cos\theta\,,$$

$$x^2 = r\sin\theta\cos\phi\,,$$

$$x^3 = r\sin\theta\sin\phi.$$

The Yang–Mills potential of the gauge group SU(2) in these coordinates are sought in the form

$$G_i^a = \frac{L^a_{ij} x^j}{r^2}\left(F(r) + 1\right) - L^a_{jk}\frac{x^i x^j x^k}{r^4}\ , \qquad (2.2)$$

where $a = 1, 2, ..., 8$; $i = 1, 2, 3$; the matrix L^a_{jk} expressed by means of the matrix generators of Lie algebra SU(3) λ^a_{jk}:

$$L^1 = \lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad L^2 = -i\lambda^2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

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$$L^3 = \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad L^4 = \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$L^5 = -i\lambda^5 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad L^6 = \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$L^7 = -i\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad L^8 = \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

The potential (2.2) in the spherical system of coordinates has the form

$$G_r^a = \Big\{ \sin 2\theta \, \cos\phi; \, \cos^2\theta - \sin^2\theta \, \cos^2\phi; \, \sin 2\theta \, \sin\phi;$$

$$0; \, \sin^2\theta \, \sin 2\phi; \, 0; \, \frac{1 - 3\sin^2\theta \, \sin^2\phi}{\sqrt{3}} \Big\} \frac{F(r)}{r}, \tag{2.3'}$$

$$G_\theta^a = \Big\{ \cos 2\theta \, \cos\phi; \, \cos\phi; \, \frac{\sin 2\theta(\sin^2\phi - 2)}{2}; \, \cos 2\theta \, \sin\phi;$$

$$\sin\phi; \, \frac{\sin 2\theta \, \sin^2\phi}{2}; \, 0; \, -\frac{\sqrt{3}}{2} \sin 2\theta \, \sin 2\phi \Big\} (F(r) + 1), \tag{2.3''}$$

$$G_\phi^a = \Big\{ -\frac{\sin 2\theta \, \sin\phi}{2}; \, -\frac{\sin 2\theta \, \sin\phi}{2}; \, \frac{\sin^2\theta \, \sin 2\phi}{2}; \, \frac{\sin 2\theta \, \cos\phi}{2};$$

$$\frac{\sin 2\theta \, \cos\phi}{2} \quad \sin^2\theta \, cos 2\phi; \, \sin^2\theta;$$

$$-\frac{\sqrt{3}}{2} \sin^2\theta \, \sin 2\phi \Big\} (F(r) + 1). \tag{2.3'''}$$

Assume that the electrical field has the only nonzero component of Maxwell's tensor:

$$F_{\mathrm{tr}} = -\frac{\partial A_t}{\partial r} = -A_t'. \tag{2.4}$$

The complete Lagrangian in this case can be written as follows:

$$L = \sqrt{-g} \left(-\frac{R}{16\pi\gamma} - \frac{1}{4e^2} W_{\mu\nu}^a W^{a\mu\nu} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right), \tag{2.5}$$

where R is a scalar curvature of the metrics (2.1), $W^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu - f^{abc} G^b_\mu G^c_\nu$ is the gauge SU(3) field strength, f^{abc} is the structure constant of the SU(3) group, e is the gauge coupling constant, and $\mu, \nu = 0, 1, 2, 3$. Within the unessential factor, Lagrangian (2.5) can be written in the form

$$L = \frac{\sigma' u}{r} + \sigma - \frac{\kappa\sigma}{r^4}\left[2u(rF' - F)^2 + 2uF^4 + \frac{r^2}{2}(F^2 - 1)^2\right]$$

$$+ 8\pi\gamma \frac{r^2}{\sigma}(A_t')^2 , \tag{2.6}$$

where $\kappa = 8\pi\gamma / e^2$. The variation (2.6) with A, σ, u, and F leads to the following equation:

$$\left(\frac{r^2}{\sigma} A_t'\right)' = 0, \tag{2.7}$$

$$\left(\frac{u}{r}\right)' = 1 - \frac{\kappa}{r^4}\left[2u(rF' - F)^2 + 2uF^4 + \frac{r^2}{2}(F^2 - 1)^2\right] - \frac{q^2}{r^2}, \tag{2.8}$$

$$\frac{\sigma'}{\sigma} = \frac{2\kappa}{r^3}[(rF' - F)^2 + F^4], \tag{2.9}$$

$$\left[\frac{u\sigma}{r^3}(rF' - F)\right]' = -\frac{\sigma u}{r^4}(rF' - F) + \frac{2\sigma u F^3}{r^4} + \frac{\sigma}{2r^2} F(F^2 - 1), \tag{2.10}$$

where q is a constant which is proportional to the electric charge.

The solution of Maxwell's equation (2.7) is

$$A_t' = \frac{\sigma q}{r^2 \sqrt{8\pi\gamma}}.$$

The function σ can be excluded from Eqs. (2.8) and (2.10) with help of (2.9). We assume that there is the event horizon at $r = r_H$, i.e., $u(r_H) = 0$. Here we introduce a dimensionless variable and the following quantity:

$$e^x = \frac{r}{r_H},\ e^{2\phi} = \frac{u}{r_H^2},\ \alpha = \frac{\kappa}{r_H^2},\ Q = \frac{q}{r_H^2}.$$

After some simplifications we can write the Einstein–Yang–Mills equation in the form

$$(F' - F)' + (F' - F)\left[e^{2x-2\phi} - \frac{\alpha(F^2-1)^2}{2}e^{-2\phi} - 1\right]$$
$$= 2F^3 + \frac{F(F^2-1)}{2}e^{2x-2\phi}, \tag{2.11}$$

$$\phi' + \frac{1}{2}\left[\frac{\alpha(F^2-1)^2}{2}e^{-2\phi} - e^{2x-2\phi}\right]$$
$$= \frac{1}{2} - \alpha e^{-2x}[(F'-F)^2 + F^4] - Q^2 e^{-2\phi}. \tag{2.12}$$

Here the prime denotes the derivative of x.

The solution

Since Eq. (2.11) is singular at the point $x = 0$, a numerical solution of the system (2.11), (2.12) at the point $x = \Delta(\Delta \ll 1)$. If the functions $\phi(x)$ and $F(x)$ are developed as a series in powers of x, we can write

$$[e^{2\phi(x=0)}]' = 1 - \frac{\alpha(F^{*2}-1)^2}{2} - Q^2, \tag{3.1}$$

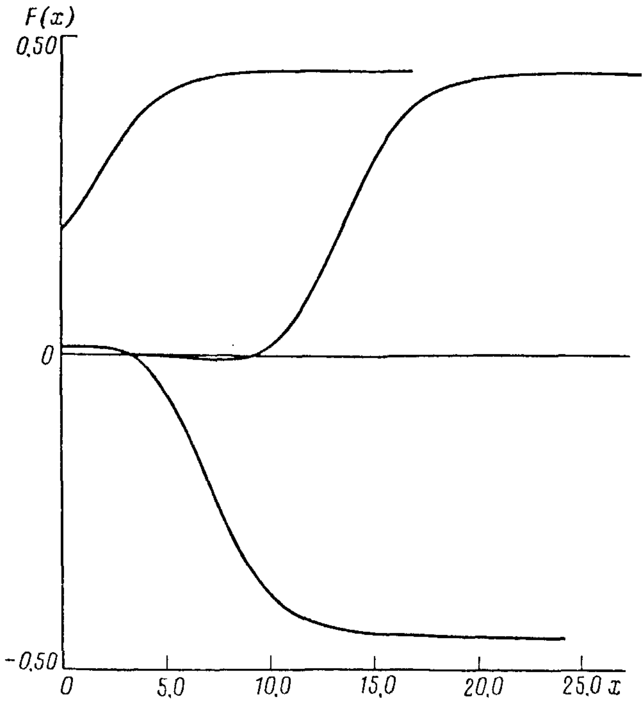

FIG. 1. The functions $V_0(x)$, $V_1(x)$, and $V_2(x)$. $\alpha = 1$, $Q = 0$.

$$F'(x=0) = F^* + \frac{F^*(F^{*2}-1)}{2\left[1 - \frac{\alpha(F^{*2}-1)^2}{2}\right]} , \tag{3.2}$$

where $F^* = F(x=0)$. Then

$$\phi(x=\Delta) \approx \frac{1}{2}\ln\left\{\Delta\left[1 - \frac{\alpha(F^{*2}-1)}{2}\right]\right\} ,$$

$$F(x=\Delta) \approx F^* + F'(x=0) .$$

The numerical solution is constructed on the basis of the Runge–Kundt–Merson method with a variable step. It was established that the solution of system (2.11), (2.12) is analogous to the solution of the SU(2) field obtained in Ref. 1. In other words, we have the regions of value F^*, in which F^* approaches either $+\infty$ or $-\infty$ as $x \to \infty$. This means that the points F^* on the boundary between these regions give the regular solution. Like in Ref. 1, this solution can be a number of integer n, which shows the extent to which the solution of $F(x)$ intersects the x axis. Figure 1 shows the functions $F_0(x)$, $F_1(x)$, and $F_2(x)$ at $Q=0$. Here $F_0^* = 0.192271...$, $F_1^* = 0.00993494...$, and $F_2^* = 0.000428997...$. The asymptotical meaning of the solutions of $F_n(x)$ is the value $F_\infty = \pm 1/\sqrt{5}$, which can be seen from (2.11). Figure 2

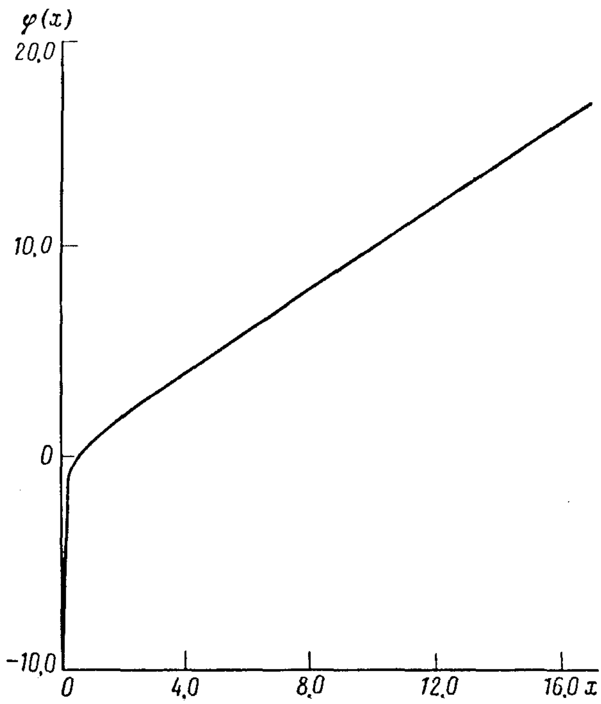

FIG. 2. The function $\phi(x)$. $\alpha = 1$, $Q = 0$.

shows the function $\phi(x)$ in which $\phi_0(x)$, $\phi_1(x)$, and $\phi_2(x)$ nearly coincide. All the solutions $\phi_n(x) \to x$ as $x \to \infty$.

The black holes which fill the chromodynamic SU(3) field were constructed in this manner.

[1]M. S. Volkov and D. V. Gal'tsov, Yad. Fiz. **51**, 1171 (1990) [Sov. J. Nucl. Phys. **51**, 747 (1990)].

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