Modification of the Stokes formula (laminar motion of a heat-evolving sphere)

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Navier–Stokes equations are written for the case of the laminar motion of a heat-evolving sphere in a liquid with a temperature-dependent viscosity. If the relative velocity of the liquid and the sphere is low, the temperature profile near the sphere is spherically symmetric. Solutions of the hydrodynamic equations with convection are derived for the case in which the logarithmic derivative of the viscosity can be assumed constant. There can also be a purely convective solution with a vanishing velocity at infinity. Only if there is absolutely no convection do the solutions and the expressions for the drag have the Stokes formula (for a constant viscosity) as a limit.

The classical Stokes formula is determined by the hydrodynamic solution at distances on the order of the radius of the sphere. That this is true can be seen from, for example, the results found by Zel'dovich:[1] In an expansion of the drag in powers of the Reynolds number Re, large distances make a correction of first order, while small distances make a correction of second order and do not alter the coefficient of the leading (Stokes) term.

For most liquids and plastic solids, the viscosity η falls off sharply with increasing temperature. This dependence can be approximated well over a broad range by

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$\eta(T) = \eta_0 \exp(\Theta/T)$, where Θ is several times the melting point (or softening point) of the material.

If the thermal power of a sphere is sufficient to increase the fluidity (the reciprocal of the viscosity) near the surface of the sphere, the hydrodynamics in this region will thus be quite different from the Stokes solution. As a result, the expression for the drag should be modified.

We might cite the solutions of the variable-viscosity problem in a one-dimensional geometry[2] and for flow in a tube.[3] The most important qualitative conclusion drawn for the nonlinear problem[2,3] is that a steady-state solution does not exist for arbitrary values of the parameters of the problem. There is the possibility in principle of a "thermal explosion," in which more heat is evolved by viscous friction that can be removed by heat conduction. In the case of a heat-conducting sphere, a thermal explosion would be possible only under conditions of little interest for the problem at hand: In three dimensions, the heat removal is more effective than in the one-or two-dimensional case.

Temperature field. A distinctive feature of the variable-viscosity problem with a spatially bounded heat source is that the heat-conduction and Navier–Stokes equations[4] can be uncoupled over broad ranges of the parameter values.

The steady-state heat-conduction equation

$$\rho c_p v_k \frac{\partial T}{\partial x_k} = \frac{\partial}{\partial x_k} \kappa(T) \frac{\partial T}{\partial x_k} \tag{1}$$

can be solved easily over distances for which the thermal conductivity can be assumed constant ($\kappa = \kappa_\infty$). This is done by replacing the velocity v_i in this equation by a constant: the relative velocity of the object and the medium, u_i. In spherical coordinates we would have

$$T(r,\theta) = T_\infty \left[1 + \frac{R_1}{r} \exp\left(-\frac{(1-\cos\theta)ur}{\chi}\right)\right], \qquad \left(R_1 = \frac{P}{4\pi\kappa_\infty T_\infty}\right). \tag{2}$$

Here P is the thermal power of a sphere of radius R; the subscript ∞ means the value far from the sphere; and $\chi = \kappa_\infty/\rho c_p$ is the thermal diffusivity.

It can be seen from the solution that at distances much smaller than the thermal length scale, $r \ll \chi/u$, the temperature profile is independent of the angle. In other words, the temperature profile is spherically symmetric. Central symmetry is a condition for the applicability of the solutions found below. [The case with the condition opposite (3), with a constant viscosity of a molten material, was studied in Ref. 5.] Since we are interested in high viscosities (large Prandtl numbers $\Pr = \eta/\rho\chi$), it follows automatically from the spherical symmetry of the temperature field that we are dealing with a laminar flow:

$$\mathrm{Re} = \frac{uR\rho}{\eta} \ll \mathrm{Re}\,\Pr = \mathrm{Pe} = \frac{uR}{\chi} \ll 1. \tag{3}$$

The relative velocity u in solution (2) and condition (3) must itself be found from the solution of the Navier–Stokes equation or, more precisely, by equating the drag to the Archimedes force. Consequently, the range of applicability of the solutions in terms of the thermal properties will be determined at the end of this letter.

A parameter of this problem which as the dimensions of a length is the quantity R_1 in (2). Below we assume that the ratio of R_1 to the sphere radius R is on the order of unity; we will allow the case $R_1/R \ll 1$, but we will exclude (in this letter) the case $R_1/R \gg 1$. When the ratio R_1/R is comparable to unity, the temperature dependence of the thermal conductivity becomes important. If this is a power-law dependence, i.e., if $\kappa(T) = \kappa_\infty (T/T_\infty)^\nu$, then a solution of the heat-conduction equation can be found analytically:

$$T|_{\nu\neq 1}(r) = T_\infty \left[1 + \frac{(1-\nu)R_1}{r}\right]^{1/(1-\nu)}, \quad T|_{\nu=1}(r) = T_\infty \exp\left(\frac{R_1}{r}\right). \tag{4}$$

We see that a thermal explosion occurs if $\nu > 1$ and if the thermal power leads to the satisfaction of the condition $R_1 > R/(\nu - 1)$. For the hydrodynamics of flow around an object, this combination is of little interest.

In the region $R < r \ll R_1 \ll u/\chi$, solutions (2) and (4) coincide: $T - T_\infty \sim r^{-1}$.

Hydrodynamics. The Navier–Stokes equation is linear at small Reynolds numbers. The viscosity η in this equation is a function of the temperature; after the heat-conduction equation is solved, the viscosity becomes a given function of the coordinates. In the hydrodynamic equations[4] we retain the convective term, which is proportional to the bulk expansion coefficient β:

$$\frac{\partial p}{\partial x_i} = \frac{\partial}{\partial x_k}\eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right) - g_i\beta\rho(T - T_\infty), \quad \frac{\partial v_k}{\partial x_k} = 0. \tag{5}$$

In the coordinate system in which the sphere is at rest, the velocity at the surface of the sphere $(r = R)$ is zero. At greater distances, this velocity is equal to the constant flow velocity u_i. We do not know the magnitude of u_i, but we do know its direction: parallel to the acceleration due to the gravity, g_i. We seek a solution in the form

$$v_i = u_i\left(\frac{1}{2}\Phi(\xi) + \int_{\xi_0}^{\xi}\Phi(\xi)d\xi\right) - \frac{1}{2}n_i\, u_k n_k\, \Phi(\xi), \qquad \xi = \ln(r/R_1). \tag{6}$$

A solution in this form automatically satisfies the continuity equation $\partial v_k/\partial x_k = 0$; it must also satisfy the boundary conditions

$$\Phi(\xi_0) = 0, \quad \Phi(\infty) = 0, \quad \int_{\xi_0}^{\infty}\Phi\, d\xi = 1, \quad \xi_0 = \ln(R/R_1). \tag{7}$$

Eliminating the gradient of the pressure p in Eq. (5) by taking the curl, we find a linear, inhomogeneous, third-order equation for the function $\Phi(\xi)$:

$$\frac{\partial^3 \Phi}{\partial \xi^3} + 2(1+L)\frac{\partial^2 \Phi}{\partial \xi^2} + (L^2 + 3L - 5 + \frac{\partial L}{\partial \xi})\frac{\partial \Phi}{\partial \xi} + (L^2 - 5L - 6 + \frac{\partial L}{\partial \xi})\Phi$$

$$= -2\mathrm{Gr}\exp(-\int_{\xi_0}^{\xi} (L-1)\, d\xi). \tag{8}$$

A corresponding equation (without convection) was derived by Mathews[6] in the course of solving the problem of the motion of an ion in liquid helium. This equation contains one constant: the Grashof number, defined here by

$$\mathrm{Gr} = g_k u_k \frac{\beta \rho P R}{4\pi \kappa_\infty \eta(R) u^2} = \pm \beta T_\infty \frac{g \rho R_1 R}{u \eta(R)}. \tag{9}$$

In addition, this equation contains one known function of the coordinates:

$$L = \frac{d \ln \eta(T(r))}{d \ln r}. \tag{10}$$

From the radial profile of the temperature in (4) we can find an explicit expression for the function L. We write it here for two values of the exponent ν in the thermal conductivity:

$$L|_{\nu=0}(\xi) = \frac{\Theta}{2T_\infty}\frac{1}{(1+\mathrm{ch}\,\xi)}, \quad L|_{\nu=1}(\xi) = \frac{\Theta}{T_\infty}\exp(-\xi - e^{-\xi}). \tag{11}$$

The choice of origin (the zero of ξ) in Eqs. (7) and (8) is arbitrary. We have put it at $r = R_1$ to simplify expressions (11). In making this choice, we put the maximum of the function $L(\xi)$ at $\xi = 0$.

The pressure in the liquid can, in general, be expressed in terms of a solution of Eq. (8):

$$p = p_\infty + \frac{\eta u_k n_k}{2r}\left[\frac{\partial^2 \Phi}{\partial \xi^2} + (3+L)\frac{\partial \Phi}{\partial \xi} + L\Phi\right]. \tag{12}$$

Model solution. Equation (8) deserves an accurate numerical study. However, our goal here is much more modest: to derive an expression for the drag. As has been mentioned previously, the drag is determined by the flow in a layer adjacent to the sphere. The logarithmic derivative of the viscosity can be assumed to remain constant along this layer, equal to its value at the surface of the sphere: $L(\xi) \simeq L(\xi_0) = L_0$. In other words, we are examining a model for the behavior of the viscosity: $\eta(r) = \eta(R)(r/R)^{L_0}$. We will derive an exact solution for this case. This model solution can be used in turn to derive an expression for the drag which is valid at large values of L_0 and for an arbitrary function $L(\xi)$. Although this model solution is approximate, it does become the Stokes formula in the limit of small L_0.

In the case $L =$ const, it is an elementary matter to solve Eq. (8):

$$\Phi = \sum_{s=0}^{3} C_s \exp(k_s(\xi - \xi_0)). \tag{13}$$

In the sum here, the index $s = 0$ means a particular solution of the inhomogeneous equation:

$$k_0 = -L_0 + 1, \quad C_0 = \mathrm{Gr}(L_0 + 4)^{-1}. \tag{14a}$$

In order to satisfy the boundary condition at infinity, we must arrange one of the following conditions: either $L_0 > 1$ or Gr $= 0$.

The other (homogeneous) solutions correspond to the three roots of the equation

$$k^3 + 2(L_0 + 1)k^2 + (L_0^2 + 3L_0 - 5)k + (L_0^2 - 5L_0 - 6) = 0.$$

One of the solutions—that determined by the largest root, k_1—increases with the radius (or, equivalently, does not fall off sufficiently slowly at $L_0 > 6$). Consequently, the coefficient of this solution must be zero: $C_1 = 0$. The two negative roots make it possible to derive a solution which satisfies boundary conditions (7):

$$k_2 = -\frac{L_0 + 1}{2} - \frac{\sqrt{L_0^2 - 2L_0 + 25}}{2}, \quad C_2 = \frac{k_2 k_3}{k_3 - k_2}\left(1 + C_0 \frac{k_0 - k_3}{k_0 k_3}\right), \tag{14b}$$

$$k_3 = -L_0 - 1, \qquad C_3 = \frac{k_2 k_3}{k_3 - k_2}\left(-1 + C_0 \frac{k_2 - k_0}{k_0 k_2}\right). \tag{14c}$$

The velocity approaches its value at infinity in proportion to a power of the radius. At large L_0, the effective thickness of the layer involved in the flow around the sphere is R/L_0, so we are justified in using the model solution for a subsequent calculation of the drag in the physical case, in which L is described by functions (11) or similar functions. In the physical solution, the square-root expression in (14) should be expanded in powers of $L_0^{-1} \ll 1$. This should be done with caution, since different numbers of terms must be retained, depending on the particular goal being sought.[1)]

At $L_0 \sim 1$, model solution (14) is of a qualitative nature in the physical case [the actual solution of Eq. (8) depends on the particular functional dependence $L(\xi)$]. Solution (14) does have the Stokes solution ($k_2 = -3$, $k_3 = -1$) in the limit $L_0 = 0$, but—this is a crucial point—in order to take this limit, it is necessary to first set the Grashof number equal to zero and only then let $L_0 \to 0$. Note that the Grashof number is not small at small velocities u. The meaning here is that the Stokes solution not only is disrupted at large flow velocities and at large distances but is also unstable at low velocities and short distances if there is a finite heat evolution.

Drag. The drag F_i is found in the standard way,[4] as the integral over the surface of the sphere of the vertical projection of the sum of the normal pressure and the viscous stress:

$$F_i u_i = \oint dS u_i n_k \left(p\delta_{ik} - \eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right)\right).$$

Substituting solution (13), (14) into this expression, using (6) and (12), and evaluating the integrals, we find

$$F=\frac{2\pi}{3}\eta(R)Ru\left[(L_0+1)k_2^2-\mathrm{Gr}\frac{(k_0^2+k_2^2)}{(L_0+4)(L_0-1)}\right],\quad (L_0>1). \tag{15}$$

We recall that k_2 is given in expression (14b), where L_0 should be calculated from (11) at the surface of the sphere, with $\xi=\xi_0=\ln(4\pi R\kappa_\infty T_\infty/P)$. The number L_0 is proportional to the ratio Θ/T_∞ and is generally not small.

At this point we equate the drag to the Archimedes force $4\pi g\Delta\rho R^3/3$ caused by the difference ($\Delta\rho$) between the densities of the sphere and medium. With the physical case in mind, we retain only the higher powers of L_0. We also substitute in expression (9) for the Grashof number, which contains u. As a result, we find the velocity at which the heat-evolving sphere sinks (if $\Delta\rho>0$) or floats up (if $\Delta\rho<0$):

$$u=\frac{2g\Delta\rho R^2}{\eta(R)L_0^3}\left[1-\beta T_\infty L_0\frac{\rho}{\Delta\rho}\frac{R_1}{R}\right]. \tag{16}$$

Limited convection. There exists a purely convective flow. In this case the velocity at infinity is $u=0$. This case arises under the physically transparent condition

$$1=L_0\beta T_\infty\frac{\rho R_1}{\Delta\rho R}\simeq L_0\beta(T_0-T_\infty)\frac{\rho}{\Delta\rho},$$

i.e., when the difference between the densities of the sphere and the medium is canceled by the decrease in the density of the medium during thermal expansion in a layer of thickness $\delta R\simeq R/L_0$. The solution itself should be sought in such a way that v_i in (6) is initially proportional to the acceleration g_i, and the integral in (7) is equal to zero, not one. However, it is possible to simply take the limit $u\to 0$ in solution (13), (14), under the assumption $C_0\sim\mathrm{Gr}\sim u^{-1}$ and $v_i\sim u_i$. As a result, we find the convective velocity field:

$$v_i=\frac{2\Delta\rho R^2}{\eta(R)L_0^3}\left[g_i\left(\frac{1}{2}\Phi_1(\xi)+\int_{\xi_0}^{\xi}\Phi_1(\xi)d\xi\right)-\frac{1}{2}n_i\,g_kn_k\,\Phi_1(\xi)\right],$$

$$\Phi_1=\exp(k_0(\xi-\xi_0))+\frac{k_2(k_0-k_3)}{k_0(k_3-k_2)}\exp(k_2(\xi-\xi_0))$$

$$+\frac{k_3(k_2-k_0)}{k_0(k_3-k_2)}\exp(k_3(\xi-\xi_0)). \tag{17}$$

The dimensional velocity coefficient is given in the limit $L_0\gg 1$, but the function $\Phi_1(\xi)$ is written in the model approximation. The singularity at $L_0=6$ here can be canceled out:

$$\Phi_1|_{L_0=6}=\exp\left(-5(\xi-\xi_0)\right)-(1+14\xi/5)\exp\left(-7(\xi-\xi_0)\right).$$

Applicability condition. Going back to condition (3), we substitute into it the result for the flow velocity, (16) (for the case of weak convection):

$$R \ll L_0 \left[\frac{\chi\eta(R)}{g\Delta\rho}\right]^{1/3} \sim L_0 \left[\frac{\eta(R)}{\eta_{16}}\right]^{1/3} \cdot 100\,\text{m}. \quad (18)$$

In the last numerical estimate here we have used $\chi \simeq 10^{-6}$ m^2/s, $\Delta\rho \simeq 1\times 10^3$ kg/m^3, and $\eta_{16} = 10^{16}$ kg/(m·s). This result tells us, in particular, that the modified Stokes formula is applicable in geophysics.

[1)] In addition, there is a cancelable singularity at $L_0 = 6$ in (13) and (14).

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