Solution of the Derrida model with rare couplings and an asymmetric distribution coupling constants; relationship with coding

D. V. Saakyan
Erevan Physics Institute, 375036, Erevan, Armenia

(Submitted 12 December 1991; resubmitted 15 January 1992)
Pis'ma Zh. Eksp. Teor. Fiz. **55**, No. 3, 198–202 (10 February 1992)

The Derrida model is solved for the case of rare couplings. The relation between constants at which a transition occurs from a spin-glass phase to a ferromagnetic phase is found.

The transmission of information is always accompanied by a noise. There exists a nonzero probability p_{ik} that the letter of index i will be sent into the letter of index k. In order to extract information with a zero error probability, it is necessary to transmit redundant information.

We assume that the pure communication consists of N numbers σ_i from a θ-valued alphabet. There exists a constant error probability $1 - m$ that the letter σ_i will be replaced by some other letter in the transmitted communication. How can the communication be coded [how can we find $\rho_i(\sigma)$, $i = 1$–Z] in order to reconstruct the original information ($\sigma_i, i = 1$–N) without any errors?

Shannon found a certain limit below which Z cannot go under the condition of an error-free decoding. The problem of constructing such codes arose.

It was suggested in Ref. 1 that in the case of an intense noise, $0 \ll m \ll 1$, a coding of this sort which is an optimum coding, according to the Shannon theorem, is given by the Derrida model[2,3] for an asymmetric distribution of coupling constants.[4]

While the spins interact in pairs in the ordinary models, e.g., the Ising model, in the Derrida model there is a simultaneous interaction of P spins ($1 \ll P \ll N$).

This approach may not be physical, but the Derrida model still occupies a special place (among spin-glass models) in that it can be solved exactly in the Parisi theory in the case of simply a one-step violation of replica symmetry. Furthermore, limiting models (or situations) are always interesting in physics. The Hamiltonian of a model with N spins $\sigma_i = \exp(i2\pi k/Q)$ $(k = 1\text{–}Q)$ with a Potts interaction is

$$H = -\sum_{(i_1 \ldots i_P)} \sum_{r=1}^{Q-1} (\tau_{i_1 \ldots i_P} \sigma_{i_1} \ldots \sigma_{i_P})^r, \tag{1}$$

where $\tau_{i_1 - i_p}$ are coupling constants with some random distribution, given by

$$\tau = \exp(i2\pi k/Q), \quad k = 1 \ldots Q.$$

In the standard Derrida model, a summation is carried out over all the P-plets (through the choice of a set of different indices $i_1\text{–}i_p$). In the case of a pure noise we would have the distribution

$$\rho(\tau(k)) = \sum_{i=1}^{Q} \frac{1}{Q} \delta_{k,i}. \tag{2}$$

A solution of model (1) with (2) shows[3] that at high temperatures the system is in a paramagnetic phase, and below a critical temperature T_c it goes into a spin-glass phase. At and below T_c, the entropy of the system is zero. In general, the form of the interaction σ^P means that only two possibilities are physically allowed: either a complete magnetization or a zero magnetization. It thus becomes a fairly simple matter to solve the Derrida model. Let us assume that during the transmission the letter σ_i is changed into σ_j with a probability p_{ij}:

$$p_{ij} = \rho_k, \quad k = (j - i)\mathrm{mod}(Q).$$

Let us consider the distribution ρ for τ:

$$\rho(\tau(k)) \equiv \rho_k = [(1 + m(Q-1))/Q]\delta_{k,Q} + \sum_{i=1}^{Q-1} \left(\frac{1-m}{Q}\right) \delta_{k,i}. \tag{3}$$

The most important property for this model is the property of gauge invariance:

$$\sigma_i \to s_i \sigma_i, \quad \tau_{i_1 \ldots i_P} \to \tau_{i_1 \ldots i_P} (s_{i_1} \ldots s_{i_P})^{-1}.$$

This property is present by virtue of the factorization of the transition matrix p_{ij}:

$$p_{ij} = \rho_k, \quad k = (j - i)\mathrm{mod}(Q).$$

In this letter we are interested in the case of rare couplings, in which P-plets are not always present in (1) and are instead present only with the probability $CP!/N^{P-1}$. The number of couplings (P-plets) is then only

$$Z = CN. \tag{4}$$

In the case of a zero noise ($m = 1$) the unique ground state of system (1), (3), (4) with $C > 1$ is the configuration

$$\sigma_i = 1. \tag{5}$$

When a nonzero noise ($m < 1$) is turned on, it becomes necessary to increase Z to some critical $Z = CN(C > 1)$ in order to make (5) the ground state of the system. A pure communication of N letters in (5) is coded in Z numbers $\tau_{i_1 - i_p}$ in such a manner that, despite the presence of the noise, it is possible to reconstruct the original information from the letters by finding the vacuum of (1).

The general case with $\sigma_i \neq 1$ can be converted into case (3), (5) by means of a gauge transformation.

The model of rare couplings, (4), was solved in Ref. 5 for the case of a symmetric distribution (and it was solved in the $Q = 2$ case in Ref. 6). Below we will be using the technique of those previous papers.

We use the replica method to calculate the free energy of model (1), (3), (4):

$$Z^n = \langle \mathrm{tr}_\sigma \exp\{ \sum_{i_1 \dots i_P} B \sum_{\alpha=1}^{n} \sum_{r=1}^{Q-1} (\tau_{i_1 \dots i_P} \sigma_{\alpha,i_1} \dots \sigma_{\alpha,i_P})^r \} \rangle. \tag{6}$$

The average in (6) is over the distribution of τ in (3). We use the expression[5]

$$\exp(B \sum_{\alpha=1}^{n} \sigma_\alpha \tau^{n_\alpha}) = A^n \sum_{r=0}^{\infty} \mu^r \sum_{\alpha_1 \dots \alpha_r} \prod_{i=1}^{r} \sigma_{\alpha_i} \tau^{n_{\alpha_i}}, \tag{7}$$

where

$$A = \left[\frac{1}{Q} \exp(B(Q-1)) + \frac{Q-1}{Q} \exp(-B) \right], \tag{8}$$

$$\mu = \frac{\exp(BQ) - 1}{\exp(BQ) + Q - 1}.$$

The summation in (7) is over various sets of α_1–α_r which are not equal to each other. Using (7) and (8), we find

$$\exp(B \sum_{r=1}^{Q-1} (\tau_{i_1 \dots i_P})^r \sum_{\alpha=1}^{n} (\sigma_{\alpha,i_1} \dots \sigma_{\alpha,i_P})^r$$

$$= A^n \sum_{s=0}^{\infty} \mu^s \sum_{\alpha_1 \dots \alpha_s} \prod_{\beta=1}^{P} \sum_{r_1 \dots r_s} \sigma^{r_1}_{\alpha_1, i_\beta} \dots \sigma^{r_s}_{\alpha_s, i_\beta} \tau^{r_1 \dots r_s}. \tag{9}$$

Adopting the mean field approximation (which is exact in the case at hand), and using the representation

$$\delta(q^{\alpha_1 \dots \alpha_r}_{i_1 \dots i_r} - \sigma^{\alpha_1}_{i_1} \dots \sigma^{\alpha_r}_{i_r}) = \frac{1}{2\pi} \int_{-i\infty}^{i\infty} dh^{\alpha_1 \dots \alpha_r}_{i_1 \dots i_r} \exp[h^{\alpha_1 \dots \alpha_r}_{i_1 \dots i_r} (q^{\alpha_1 \dots \alpha_r}_{i_1 \dots i_r} - \sigma^{\alpha_1}_{i_1} \dots \sigma^{\alpha_r}_{i_P})],$$

of the δ-function, we find

$$-nBF = nC\ln A + C\sum_{s=1}^{\infty}\mu^s \sum_{\alpha_1\ldots\alpha_s}\sum_{r_1\ldots r_s}(q_{\alpha_1\ldots\alpha_s}^{r_1\ldots r_s})^P\langle\tau^{r_1+\ldots r_s}\rangle + \tag{10}$$

$$-\sum_{\alpha_1\ldots\alpha_s}\sum_{r_1\ldots r_s} h_{\alpha_1\ldots\alpha_s}^{r_1\ldots r_s} q_{\alpha_1\ldots\alpha_s}^{r_1\ldots r_s} + \ln \mathrm{Tr}_\sigma \exp\Big(\sum_{s=1}^{\infty}\sum_{r_1\ldots r_s}\sum_{\alpha_1\ldots\alpha_s} h_{\alpha_1\ldots\alpha_s}^{r_1\ldots r_s}\sigma_{\alpha_1}^{r_1}\ldots\sigma_{\alpha_s}^{r_s}\Big).$$

The values of the correlation functions q and the Lagrange multipliers h are found from the condition for an extremum of (10) (Refs. 3–5). We need to distinguish the even distributions $(r_1+\ldots r_s=0,\ \mathrm{mod}(Q))$ from the odd ones $(r_1+\ldots r_s\neq 0,\ \mathrm{mod}(Q))$ in (10). For even distributions we have

$$\langle\tau^{r_1+\ldots r_s}\rangle = 1. \tag{11}$$

For odd ones we have

$$\langle\tau^{r_1+\ldots r_s}\rangle = m. \tag{12}$$

We first consider the case in which replica symmetry is preserved:

$$q_{\alpha_1\ldots\alpha_s}^{r_1\ldots r_s} = q^{r_1\ldots r_s}, \quad h_{\alpha_1\ldots\alpha_s}^{r_1\ldots r_s} = h^{r_1\ldots r_s}. \tag{13}$$

From the condition for an extremum of q we have

$$h^{r_1\ldots r_s} = C\mu^s P(q^{r_1\ldots r_s})^{P-1}. \tag{14}$$

From (14) we find $q<1$, $h\to 0$ for the paramagnetic phase. Expanding $\exp(\Sigma h\sigma)$ in a series in h, and differentiating (10) with respect to h, we find

$$h^{r_1\ldots r_s} = 0, \quad q^{r_1\ldots r_s} = 0, \tag{15}$$

$$-BF = C\ln\int\Big[\frac{1}{Q}\exp(B(Q-1)) + \frac{Q-1}{Q}\exp(-B)\Big] + \ln Q.$$

For the ferromagnetic phase we have

$$q^{r_1\ldots r_s} = 1, \quad h^{r_1\ldots r_s}\to\infty. \tag{16}$$

To calculate F, we need to find the number (E_s) of even distributions and the number (O_s) of odd distributions of the numbers r_1–r_s, $r_\alpha = 1$ to $(Q-1)$. We find $O_1 = Q-1$, $E_1 = 0$. It is easy to find the recurrence relation

$$E_{n+1} = O_n, \quad E_n + O_n = (Q-1)^n. \tag{17}$$

Hence we find the equation

$$E_{n+1} = (Q-1)^n - E_n, \tag{18}$$

whose solution is

$$E_n = \frac{(Q-1)^n + (Q-1)(-1)^n}{Q}, \quad O_n = \frac{(Q-1)^{n+1} - (Q-1)(-1)^n}{Q}. \tag{19}$$

Using (19) in the case

$$q^{r_1 \ldots r_s} = 1, \quad h^{r_1 \ldots r_s} \to \infty, \tag{20}$$

we find

$$\begin{aligned} -BF = C \ln \frac{\exp(B(Q-1)) + (Q-1)\exp(-B)}{Q} \\ + \frac{1+m(Q-1)}{Q} \ln \frac{Q\exp(BQ)}{\exp(BQ)+Q-1} \\ + \frac{(Q-1)(1-m)}{Q} \ln \frac{Q}{\exp(BQ)+Q-1} = mBQ. \end{aligned} \tag{21}$$

We turn now to the case in which replica symmetry is violated. It turns out that in our case the solution of model (1), (3) is the same as the solution found in Ref. 5 [model (1) with (2)]. The reason is that $q^{r_1 \ldots r_s}$ disappears in the case of odd distributions, while in the case of even distributions we have

$$\langle \tau^{r_1 + \ldots r_s} \rangle = 1, \tag{22}$$

for both (2) and (3).

Here is the result[5] for this case ($C > 1$):

$$-F = \frac{1}{B_c} \ln Q + \frac{C}{B_c} \ln \left[\frac{1}{Q} \exp(B_c(Q-1)) + \frac{Q-1}{Q} \exp(-B_c) \right]. \tag{23}$$

The quantity B_c in (23) is found from the condition for the vanishing of the entropy in (15):

$$C = \frac{\ln Q}{\frac{B_c(Q-1)\exp(B_c(Q-1)) - \exp(-B_c)}{\exp(B_c(Q-1)) + (Q-1)\exp(-B_c)} - \ln \frac{\exp(B_c(Q-1)) + (Q-1)\exp(-B_c)}{Q}}, \tag{24}$$

$$m = \frac{\exp(B_c(Q-1)) - \exp(-B_c)}{\exp(B_c(Q-1)) + (Q-1)\exp(-B_c)}. \tag{25}$$

Finding B_c from (25), and substituting the result into (24), we easily find, for $Z = CN$,

$$Z \left[\ln Q + \frac{1+m(Q-1)}{Q} \ln \frac{1+m(Q-1)}{Q} + \frac{(Q-1)(1-m)}{Q} \ln \frac{1-m}{Q} \right] = N \ln Q. \tag{26}$$

This equation solves the problem formulated at the beginning of this letter. For a given value of m (the probability for an error is $1-m$), a necessary condition for total magnetization in the vacuum state is that the number of couplings, Z, be larger than that given by (26). On the other hand, the Shannon theorem for optimum coding,

$$\frac{Z}{N} \geq \left[\ln Q + \sum_{i=1}^{Q} p_i \ln p_i \right], \tag{27}$$

tells us that Eq. (26) gives us the limit of the best possible coding for our distribution of τ, i.e., (3). With a different distribution of τ, it would become necessary to consider a different type of interaction in Hamiltonian (1).

We could look at (27) from a different point of view. If we wish to create a spin-glass state in a system of Q-valued spins, we need an entropy for the distribution of τ (of couplings) which is greater than the threshold value $N \ln Q$. Not every type of noise would be capable of disrupting the ferromagnetic vacuum to the extent that there would be a transition to a spin-glass state. This comment applies to all spin-glass models.

It can be concluded that there is a profound relationship between the spin-glass model which we have been discussing here and coding theory. It would be interesting to exploit this interesting circumstance to make progress in both spin-glass theory and coding theory. It may be that this relationship is pertinent to the properties of ultrametricity, which is encountered in both theories.

[1]N. Sourlas, Nature **339**, 693 (1989).
[2]B. Derrida, Phys. Rev. Lett. **45**, 79 (1980).
[3]D. Gross and M. Mezard, Nucl. Phys. B **240**, 141 (1984).
[4]D. B. Saakyan, Zh. Eksp. Teor. Fiz. **100**, 236 (1991) [Sov. Phys. JETP **73**, 136 (1991)].
[5]Y. Goldschmidt, J. Phys. A **22**, L157 (1989).
[6]C. Dominicis and P. Mottishaw, J. Phys. A **20**, L1267 (1987).

Translated by D. Parsons