Conformal theory of quantum gravity from integrable hierarchies

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We propose a direct derivation of a conformal field theory description of 2D quantum gravity + matter from the formalism of integrable hierarchies subjected to Virasoro constraints. The construction is based on a generalization of the Kontsevich parametrization of the Kadomtsev–Petiashvili (KP) times achieved by introducing Miwa parameters into it.

1. Introduction. The Matrix Models,[1–3] in addition to their applications to matter + gravity systems,[4–7] have also shown intriguing relations to integrable hierarchies subjected to Virasoro constraints,[8–11] as well as to the intersection theory on the moduli space of curves.[7,12,13] However, a challenging problem of giving a direct proof of the equivalence between the 'hierarchical' formalism and the conformal field theory description of quantum gravity remains.[14–16] Another major task is to find the general solution to the Virasoro constraints on integrable hierarchies.

In this paper we show that these two problems are solved simultaneously, as the Virasoro constraints are in fact *solved* by a certain conformal theory. Our main tool will be a generalization of the Kontsevich parametrization[12] (see also Refs. 19 and 20) of the KP times obtained by introducing into it parameters of the Miwa transformation[17,18] known from the KP hierarchy. It turns out that one must allow the Miwa parameters to vary in order to be able to move between *different* (generalized) Kontsevich transformations: we will see that different Kontsevich transformations should be used, depending on the operators one considers, which we call the Kontsevich–Miwa transform.

2. Virasoro action on the KP hierarchy. The KP hierarchy equations are imposed on the coefficients $\omega_n(x,t_1,t_2,t_3,...)$ of a ψDiff operator[22] K of the form (with $D=\partial/\partial x$)

$$K = 1 + \sum_{n\geq 1} w_n D^{-n}. \tag{1}$$

The wave function and the adjoint wave function are defined by

$$\psi(t,z) = Ke^{\xi(t,z)}, \quad \psi^*(t,z) = K^{*-1}e^{-\xi(t,z)}, \quad \xi(t,z) = \sum_{r\geq 1} t_r z^r, \tag{2}$$

where K^* is the formal adjoint of K. The wave functions are related to the τ-function by

$$\psi(t,z) = e^{\xi(t,z)}\frac{\tau(t-[z^{-1}])}{\tau(t)}, \quad \psi^*(t,z) = e^{-\xi(t,z)}\frac{\tau(t+[z^{-1}])}{\tau(t)}, \tag{3}$$

where $t \pm [z^{-1}] = (t_1 \pm z^{-1}$, and $t_2 \pm \frac{1}{2}z^{-2}$, $t_3 \pm \frac{1}{3}z^{-3},\ldots)$.

The Virasoro action on the τ-function is implemented by the generators

$$\begin{aligned} \hat{L}_{p>0} &= \tfrac{1}{2}\textstyle\sum_{k=1}^{p-1}\frac{\partial^2}{\partial t_{p-k}\partial t_k} + \sum_{k\geq 1} k t_k \frac{\partial}{\partial t_{p+k}} + \left(a_0 + (J-\tfrac{1}{2})p\right)\frac{\partial}{\partial t_p}, \\ \hat{L}_0 &= \textstyle\sum_{k\geq 1} k t_k \frac{\partial}{\partial t_k} + \frac{1}{2}a_0^2 - \frac{1}{2}\left(J-\frac{1}{2}\right)^2, \\ \hat{L}_{p<0} &= \textstyle\sum_{k\geq 1}(k-p)t_{k-p}\frac{\partial}{\partial t_k} + \frac{1}{2}\sum_{k=1}^{-p-1} k(-p-k)t_k t_{-p-k} \\ &+ \left(a_0 + (J-\tfrac{1}{2})p\right)(-p)t_{-p}, \end{aligned} \tag{4}$$

which satisfy the Virasoro algebra with central charge $-12(J^2 - J + \frac{1}{6})$.

3. Miwa–Kontsevich transform. The Miwa reparametrization of the KP times is accomplished by the substitution

$$t_r = \frac{1}{r}\sum_j n_j z_j^{-r}, \tag{5}$$

where $\{z_j\}$ is a set of points on the complex plane. By the Kontsevich transform we understand the dependence, via Eq. (5), of t_r on the z_j for *fixed* n_j. To recast the Virasoro constraints $\hat{L}_{n\geq -1}\tau = 0$ into the Kontsevich parametrization, note that choosing the involved $\hat{L}$'s amounts to retaining in the energy-momentum tensor $\hat{T}(z) = \Sigma_{p\in Z} z^{-p-2}\hat{L}_p$ only terms with z to negative powers:

$$\hat{T}^{(\infty)}(v) = \sum_{n\geq 0} v^{-n-1}\frac{1}{2\pi i}\oint dz z^n \hat{T}(z) = \frac{1}{2\pi i}\oint dz \frac{1}{v-z}\hat{T}(z), \tag{6}$$

where v is taken from a neighborhood of the infinity and the integration contour encompasses this neighborhood.

A crucial simplification is achieved by evaluating $\hat{T}^{(\infty)}(v)$ only at a point from the above set $\{z_j\}$ (one must make sure that they lie in the chosen neighborhood). We must express the Virasoro action on the τ-function $\partial/\partial z_j$ derivatives, but the equation relating t_r and z_j does not allow us to substitute $\partial/\partial t_r$ in terms of $\partial/\partial z_j$. It is only after we evaluate the residues in (6) that we find the t-derivatives arranged into the combinations which are just the desired $\partial/\partial z_j$'s, aside from the term $-(J - \frac{1}{2} - 1/2n_i)\Sigma_{r\geq 1} r z^{-r-2}\partial/\partial t_r$ which should thus be set to zero by choosing

$$n_i = \frac{1}{2J-1} \equiv \frac{1}{Q}. \tag{7}$$

We thus obtain the action of $\hat{T}^{\infty}(z_i)$ on the τ-function in the Kontsevich parametrization given by the operator

$$\mathcal{T}_{\{n\}}(z_i) = -\frac{Q^2}{2}\frac{\partial^2}{\partial z_i^2} - \sum_{j\neq i}\frac{1}{z_j - z_i}\left(\frac{\partial}{\partial z_j} - Qn_j\frac{\partial}{\partial z_i}\right). \tag{8}$$

This operator depends on the collection of the n_j with $j\neq i$. In particular, if one wishes *all* the $\widehat{T}^{(\infty)}(z_j)$ to carry over to the Kontsevich variables along with $\widehat{T}^{(\infty)}(z_i)$, all the n_j must be set to the same value in (7). One then obtains the "symmetric" operators

$$\mathcal{T}(z_i) = -\frac{Q^2}{2}\frac{\partial^2}{\partial z_i^2} - \sum_{j\neq i}\frac{1}{z_j - z_i}\left(\frac{\partial}{\partial z_j} - \frac{\partial}{\partial z_i}\right). \tag{9}$$

These operators satisfy the centerless algebra spanned by the $\{\hat{L}_{n\geqslant -1}\}$ Virasoro generators. Then, if one starts with the *Virasoro-constrained* KP hierarchy, i.e., $\widehat{T}^{(\infty)}(z)\tau = 0$, one ends up with the KP Virasoro *master equation* (cf. Ref. 20) $\tau(z_i)\cdot\tau\{z_j\} = 0$.

5. Conformal field theory from Virasoro constraints. We now introduce a conformal theory of a $U(1)$ current $j(z) = \Sigma_{n\in Z}\, j_n z^{-n-1}$ and an energy-momentum tensor $T(z) = \Sigma_{n\in Z} L_n z^{-n-2}$:

$$\begin{aligned}
[j_m, \quad j_n] &= km\delta_{m+n,0} \\
[L_m, \quad L_n] &= (m-n)L_{m+n} + \tfrac{d+1}{12}(m^3 - m)\delta_{m+n,0}\,. \\
[L_m, \quad j_n] &= -nj_{m+n}
\end{aligned} \tag{10}$$

(We have parametrized the central charge as $d+1$.) Let Ψ be a primary field with a conformal dimension Δ and $U(1)$ charge q. Then, in the standard setting of Ref. 21, we find that the level-2 vector

$$|\Upsilon\rangle = \left(\alpha L_{-1}^2 + L_{-2} + \beta j_{-2} + \gamma j_{-1}L_{-1}\right)|\Psi\rangle \tag{11}$$

is primary provided that

$$\alpha = \frac{k}{2q^2}\,, \quad \beta = -\frac{q}{k} - \frac{1}{2q}\,, \quad \gamma = -\frac{1}{q}\,, \quad \Delta = -\frac{q^2}{k} - \frac{1}{2}\,, \tag{12}$$

with q and Δ given by

$$\frac{q^2}{k} = \frac{d - 13 \pm \sqrt{(25-d)(1-d)}}{24}, \quad \Delta = \frac{1 - d \mp \sqrt{(25-d)(1-d)}}{24}\,. \tag{13}$$

Factoring out the state (11) leads in the usual manner to the equation

$$\left\{\frac{k}{2q^2}\frac{\partial^2}{\partial z^2} - \frac{1}{q}\sum_j \frac{1}{z_j - z}\left(q\frac{\partial}{\partial z_j} - q_j\frac{\partial}{\partial z}\right)\right.$$

$$\left. + \frac{1}{q}\sum_j \frac{q\Delta_j - q_j\Delta}{(z_j - z)^2}\right\}\langle\Psi(z)\Psi_1(z_1)\dots\Psi_n(z_n)\rangle = 0\,, \tag{14}$$

where Ψ_j are primaries of dimension Δ_j and $U(1)$ charge q_j. In particular,

$$\left\{\frac{k}{2q^2}\frac{\partial^2}{\partial z_i^2}+\sum_{j\neq i}\frac{1}{z_i-z_j}\left(\frac{\partial}{\partial z_j}-\frac{\partial}{\partial z_i}\right)\right\}\langle\Psi(z_1)\dots\Psi(z_n)\rangle=0. \qquad (15)$$

Writing the Hilbert space as (matter) $\otimes$ (current) $= M\otimes C$, $|\Psi\rangle=|\psi\rangle\otimes|\widetilde{\Psi}\rangle$, we introduce the matter Virasoro generators l_n by

$$L_n=l_n+\tilde{L}_n\equiv l_n+\frac{1}{2k}\sum_{m\in\mathbf{Z}}:j_{n-m}j_m. \qquad (16)$$

They will then have the central charge d. It turns out that

$$|\Upsilon\rangle=\left(\frac{k}{2q^2}l_{-1}^2+l_{-2}\right)|\Psi\rangle, \qquad (17)$$

and thus we are left with a null vector in the matter Hilbert space M. Now, the dimension of $|\psi\rangle$ in the matter sector,

$$\delta=\Delta-\frac{1}{2k}q^2=\frac{5-d\mp\sqrt{(25-d)(1-d)}}{16}, \qquad (18)$$

is (for the appropriate values of d) that of the '21' operator of the minimal model with central charge d.[24,25]

The above can now be used to solve the Virasoro constraints on the KP hierarchy by assuming the ansatz[23]

$$\tau\{z_j\}=\lim_{n\to\infty}\langle\Psi(z_1)\dots\Psi(z_n)\rangle. \qquad (19)$$

Comparing Eqs. (14) and (8), we then find

$$Q^2=-\frac{k}{q^2}=\frac{13-d\pm\sqrt{(25-d)(1-d)}}{6}, \qquad (20)$$

and d is therefore determined in terms of the parameter Q from (4) [where $J=(Q+1/2)$], as $d=13-3Q^2-12/Q^2$.

To see what are the matter theory field operators, which can be derived from the Virasoro constraints, consider the form the $L_{n\geqslant -1}$ constraints take for the wave function of the hierarchy, $w(t,z_k)\equiv e^{-\xi(t,z_k)}\psi(t,z_k)$, which should now become a function of the z_j, $w\{z_j\}(z_k)$. More precisely, consider the 'unnormalized' wave function $\overline{w}\{z_j\}(z_k)=\tau\{z_j\}w\{z_k\}$. Then the use of the Kontsevich transform at the Miwa point $n_j=1/Q$, $j\neq k$ and $n_k=-1$, gives[1)]

$$\bar{w}\{z_j\}(z_k)=\left\langle\prod_{j\neq k}\Psi(z_j)\cdot\Xi(z_k)\right\rangle, \qquad (21)$$

where Ξ is a primary field with the $U(1)$ charge qQ and dimension $Q\Delta$. Now we choose in (20) the branch of the square root $\sqrt{Q^2}$, so that Q can be positive for $d<1$:

$$Q=\frac{1}{2}\sqrt{\frac{25-d}{3}}\pm\frac{1}{2}\sqrt{\frac{1-d}{3}}\equiv-\frac{-Q_{\rm L}\mp Q_{\rm m}}{2}\equiv-\alpha_{\mp} \tag{22}$$

[with the upper/lower signs corresponding to those in (20)]. This establishes the physical meaning of the background charge Q which is present initially in the Virasoro constraints. [Note that this charge has entered explicitly in the Kontsevich transform through (7)]. Now the dimension of Ξ is equal to $\mp\frac{1}{2}Q_m$, which implies in turn that its dimension in the matter sector is

$$\mp\frac{1}{2}Q_{\rm m}-\frac{1}{2k}(qQ)^2=\mp\frac{1}{2}Q_{\rm m}+\frac{1}{2}\equiv\left\{\begin{array}{l}1-J_{\rm m}\\ J_{\rm m}\end{array}\right., \tag{23}$$

where J_m is the conformal 'spin' (dimension) of a bc system. Thus, the wave function is associated (for, say, the lower signs) with the b-field B of a bc system. The adjoint wave function will then be similarly related to the corresponding c field C: for instance, the function $\tau(t-[z_k^{-1}]+[z_l^{-1}])$ is proportional to the correlation function[2)]

$$\left\langle\prod_{\substack{j\neq k\\ j\neq l}}\Psi(z_j)\exp\left(\frac{qQ}{k}\int^{z_k}j\right)B(z_k)\exp\left(-\frac{qQ}{k}\int^{z_l}j\right)C(z_l)\right\rangle$$

$$=\left\langle\prod_{\substack{j\neq k\\ j\neq l}}\Psi(z_j)(z_k-z_l)\exp\left(\frac{qQ}{k}\int_{z_l}^{z_k}j\right)B(z_k)C(z_l)\right\rangle. \tag{24}$$

Note that this is *not* the system of free fermions underlying the construction of the general (i.e., not Virasoro-constrained) τ-functions. By bosonization one obtains a scalar φ with the energy-momentum tensor $T_m=-\frac{1}{2}\partial\varphi\partial\varphi+\frac{i}{2}Q_m\partial^2\varphi$, thus establishing the relation with minimal models[21,24,25] (for appropriate values of the central charge $d=1-3Q_m^2$).

Further, as to the theory in C, recall that we have $[j_m,j_n]=km\times\delta_{m+n,0}$, $j_{n>0}|\Psi\rangle=0$, $j_0|\Psi\rangle=q|\Psi\rangle$ with negative q^2/k (for $d<1$). To see what the current corresponds to in the KP theory, consider the correlation function with an extra insertion of an operator which depends only on j:

$$\left\langle\prod_{j\neq k,\ j\neq l}\Psi(z_j)\exp\frac{2\Delta}{q}\int_{z_l}^{z_k}j\right\rangle. \tag{25}$$

The decoupling equation states that the correlation function (25) coincides, within a constant, with the τ-function $\tau(t)$ which is evaluated at the Miwa point $n_j=1/Q$, $j\neq k, j\neq l$, $n_k=Q-(2/Q)$, $n_l=(2/Q)-Q$. Here $|Q-(2/Q)|=Q_m$; using the notation, the function we are considering can be written as $\tau\{z_j\}_{\substack{j\neq k\\ j\neq l}}(t-Q_m[z_k^{-1}]+Q_m[z_l^{-1}])$.

The balance of dimensions and $U(1)$ charges of the Ψ and Ξ operators follows a particular pattern: we find from (14) that $\Delta_j = \Delta(q_j/q)$. The dimension in the matter sector M will then be

$$\delta_j = \Delta_j - \frac{q_j^2}{2k} = \Delta\frac{q_j}{q} - \frac{q_j^2}{2k}. \tag{26}$$

Since the coefficient at the term linear in $q_j/\sqrt{-k}$ is $\frac{1}{2}Q_m$, this equation will always be satisfied for the matter operators $e^{i\gamma\varphi}$, provided that $q_j/\sqrt{-k} = \gamma$! Thus the 'dressing' prescription inherited from the KP hierarchy shows that the coefficients of the two scalars φ and ϕ that enter the exponents coincide (to be precise, within the factor of i). Therefore, although the field content is the same as in Ref. 15, the David–Distler–Kawai formalism is not recovered directly from the KP hierarchy.

The 'bulk' dimensions Δ_j, rather than being equal to 1, are related to the gravitational scaling dimensions of the fields. Evaluating the gravitational scaling dimension of ψ according to Refs. 14–16,

$$\hat{\delta}_\pm = \frac{\pm\sqrt{1-d+24\delta} - \sqrt{1-d}}{\sqrt{25-d} - \sqrt{1-d}}, \tag{27}$$

we find

$$\hat{\delta}_+ = \frac{3}{8} \pm \frac{d-4-\sqrt{(1-d)(25-d)}}{24}, \tag{28}$$

with the sign on the RHS corresponding to that in (13) and the subsequent formulas. In particular, choosing the *lower* signs throughout, we have $\hat{\delta}_+ = \Delta + \frac{1}{2}$. More generally, the gravitational scaling dimensions corresponding to (26) are

$$\hat{\delta}_{j+} = -\frac{q_j q}{k} = \Delta_j + \frac{1}{2}\frac{q_j}{q} = \Delta_j + \frac{1}{2}Q\frac{q_j}{\sqrt{-k}}. \tag{29}$$

(Again, this is valid for the '+' gravitational scaling dimensions and for the lower signs in Eqs. (20), etc.; i.e., for only one out of four possibilities to choose the signs.)

5. Concluding remarks. 1. Various aspects of the conversion of Virasoro constraints into decoupling equations deserve more study from the 'Liouville' point of view. The Kontsevich-type matrix integral, whose Ward identifies coincide with our master equation, may thus provide a candidate for a discretized definition of the Liouville theory.

2. For the matter central charge d from the minimal-models series, how can the *higher* null-vector decoupling equations be obtained starting from the Virasoro-constrained hierarchies?

[1)]To obtain the insertion into the correlation function (21) at the point z_k of the operator of interest to us, rather than its fusion with the 'background' Ψ, we use the Kontsevich transform at the value of the Miwa parameter $n_k = -1$, instead of $1/Q - 1$. This means that we are in fact considering $\overline{w}\{z_j\}_{j\neq k}(z_k)$. Similar remarks apply to other correlation functions considered below.

[2)]Therefore, the whole 'Borel' subalgebra of the W_∞ algebra,[26] which is the symmetry algebra fo the Viarasoro-constrained KP hierarchy,[27] is represented in terms of the bilocal operator insertions, reckoned from (24), which are placed at the points from the *fixed set* $\{z_j\}\times\{z_j'\}\subset CP^1\times CP^1$.

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