Symmetry group of the kinetic equations for a collisionless plasma

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A continuous Lie point group is found to be allowed by the system of integrodifferential Vlasov equations in the theory of an electron–ion plasma with a self-consistent electromagnetic field. Under the conditions that the charges and masses of the plasma particles as well as the velocity of light in vacuum are invariant, this group is a finite 12-parameter group. It contains as a subgroup the Poincaré 10-parameter group. When the renormalization of the constants of the theory is taken into account, this group becomes infinite, with an arbitrariness in 16 functions of five variables.

A hot plasma is a quasineutral gas of charged particles with a negligibly low rate of Coulomb collisions. The theory for such plasmas is based on the system of Vlasov kinetic equations,[1]

$$\frac{\partial f}{\partial t}+\vec{V}\frac{\partial f}{\partial \vec{r}}+e\left\{\vec{E}+\frac{1}{c}[\vec{V},\vec{B}]\right\}\frac{\partial f}{\partial \vec{p}}=0;\quad \vec{p}=\frac{m\vec{V}}{\sqrt{1-(\vec{V}/c)^2}};$$

$$\frac{\partial \bar{f}}{\partial t}+\vec{V}\frac{\partial \bar{f}}{\partial \vec{r}}+\bar{e}\left\{\vec{E}+\frac{1}{c}[\vec{V},\vec{B}]\right\}\frac{\partial \bar{f}}{\partial \vec{q}}=0;\quad \vec{q}=\frac{\bar{m}\vec{V}}{\sqrt{1-(\vec{V}/c)^2}}, \tag{1}$$

for the distribution functions f and $\bar{f}$ of the plasma particles, which are moving in a self-consistent electromagnetic field. The field vectors of this field, $\vec{E}$ and $\vec{B}$, obey Maxwell's equations

$$c\,\mathrm{curl}\,\vec{E} + \frac{\partial \vec{B}}{\partial t} = 0; \quad \mathrm{div}\vec{E} = 4\pi\rho;$$

$$c\,\mathrm{curl}\,\vec{B} - \frac{\partial \vec{E}}{\partial t} = 4\pi\vec{j}; \quad \mathrm{div}\vec{B} = 0. \tag{2}$$

The charge density ρ and the current density $\vec{j}$ are in turn governed by the motion of the particles:

$$\rho = e\int d\vec{p}f + \bar{e}\int d\vec{q}\bar{f}; \quad \vec{j} = e\int d\vec{p}\vec{V}f + \bar{e}\int d\vec{q}\vec{V}\bar{f}. \tag{3}$$

Equations (1)–(3) have broad applications in physical kinetics. In particular, they play an important role in the theory for the plasmas of interest to the effort to achieve controlled fusion.

In the present letter we are reporting a direct calculation of the coordinates of an infinitesimal operator of a continuous Lie point group which is allowed by system (1)–(3). The calculation is carried out by the standard Lie–Ovsyannikov–Ibragimov group-analysis scheme[2-4] for systems of partial differential equations, supplemented in a substantial way by elements of the approach of Ref. 5. That approach makes it possible to deal with nonlocal contributions (3) in this scheme. In addition, working from the basic idea of the renormalization-group method,[6,7] we include in the set of variables subject to the group transformation [the variables in (4)] five parameters of system (1)–(3). These parameters are the charges e, $\bar{e}$ and the masses m, $\bar{m}$ of the plasma particles (there are two species of particles) and the velocity of light, c:

$$\{t, x_i, V_i, e, m, \bar{e}, \bar{m}, c; f, \bar{f}, E_i, B_i, j_i, \rho\}. \tag{4}$$

Here the x_i are the components of the radius vector $\vec{r}$. The index i takes on three values, in accordance with the dimensionality of the vectors in (1)–(3). The infinitesimal operator which arises in this group analysis has an arbitrariness in 16 independent scalar functions of five parameters. It can be represented by 16 infinitesimal operators with coefficients in the form of these functions:

$$A_0, A_i, b_i, g_i, A_4, A_5, \ldots A_9 - \text{functions of} \quad e, m, \bar{e}, \bar{m}, c. \tag{5}$$

The Poincaré group, which is obvious from physical considerations, corresponds to a subgroup represented here by ten (scalar) infinitesimal operators:

$$X_0 = A_0\frac{\partial}{\partial t}; \; X_i = A_i\frac{\partial}{\partial x_i}; \; Y_i = b_i\left[x_i\frac{\partial}{\partial t} + c^2 t\frac{\partial}{\partial x_i} + (c^2\delta_{is} - V_i V_s)\frac{\partial}{\partial V_s}\right.$$
$$\left. - c e_{isk} B_s \frac{\partial}{\partial E_k} + c e_{isk} E_s \frac{\partial}{\partial B_k} + c^2\rho\frac{\partial}{\partial j_i} + j_i\frac{\partial}{\partial \rho}\right];$$
$$Z_i = g_i e_{isk}\left[x_s\frac{\partial}{\partial x_k} + V_s\frac{\partial}{\partial V_k} + E_s\frac{\partial}{\partial E_k} + B_s\frac{\partial}{\partial B_k} + j_s\frac{\partial}{\partial j_k}\right]. \tag{6}$$

Here and below, a summation is to be carried out over the indices s and k, but not over i; and e_{isk} is the Levi–Civita density. This subgroup, (6), is supplemented with infinitesimal operators corresponding to extensions and translations:

$$X_4 = A_4 \left\{ t\frac{\partial}{\partial t} + x_s\frac{\partial}{\partial x_s} - 2f\frac{\partial}{\partial f} - 2\bar{f}\frac{\partial}{\partial \bar{f}} - E_s\frac{\partial}{\partial E_s} - B_s\frac{\partial}{\partial B_s} - 2j_s\frac{\partial}{\partial j_s} - 2\rho\frac{\partial}{\partial \rho} \right\};$$

$$X_5 = A_5 \left\{ \frac{1}{m^3 e}\frac{\partial}{\partial f} - \frac{1}{\bar{m}^3 \bar{e}}\frac{\partial}{\partial \bar{f}} \right\}. \tag{7}$$

If the parameters of the theory are invariant, the functions in (5) are constants, and the continuous Lie group allowed by Eqs. (1)–(3), i.e., (6), (7), is a finite 12-parameter group. In general, the set of infinitesimal operators in (6), (7) is supplemented by four more infinitesimal operators. They appear because of the group transformation of the velocity of light and the charges and masses of the plasma particles:

$$X_6 = A_6 \left[c\frac{\partial}{\partial c} + x_s\frac{\partial}{\partial x_s} + V_s\frac{\partial}{\partial V_s} - 3f\frac{\partial}{\partial f} - 3\bar{f}\frac{\partial}{\partial \bar{f}} + E_s\frac{\partial}{\partial E_s} + B_s\frac{\partial}{\partial B_s} + j_s\frac{\partial}{\partial j_s} \right];$$

$$X_7 = A_7 \left[m\frac{\partial}{\partial m} + \bar{m}\frac{\partial}{\partial \bar{m}} - 2f\frac{\partial}{\partial f} - 2\bar{f}\frac{\partial}{\partial \bar{f}} + E_s\frac{\partial}{\partial E_s} + B_s\frac{\partial}{\partial B_s} + j_s\frac{\partial}{\partial j_s} + \rho\frac{\partial}{\partial \rho} \right];$$

$$X_8 = A_8 \left[\bar{m}\frac{\partial}{\partial \bar{m}} + \bar{e}\frac{\partial}{\partial \bar{e}} - 4\bar{f}\frac{\partial}{\partial \bar{f}} \right]; \quad X_9 = A_9 \left[m\frac{\partial}{\partial m} + e\frac{\partial}{\partial e} - 4f\frac{\partial}{\partial f} \right]. \tag{8}$$

The 16 scalar infinitesimal operators in (6)–(8) form a Lie algebra, aside from the arbitrariness in the functions in (5). These operators constitute an infinite continuous group which is allowed by the system of Vlasov equations in (1)–(3).

Equations (6)–(8) constitute the basic result of this letter. Let us examine this result. The appearance of the hyperbolic rotations Y_i is a consequence of two circumstances: the dependence of the momenta of the plasma particles on the velocity and the "relativistic" nature of the field equations in (2). If one of these functional dependences is changed (e.g., in the case of quasiparticles in the crystal lattice of a metal), the group changes. In particular, the Galilean group corresponds to the following approximate expression for the infinitesimal operators Y_i:

$$Y_i \approx c^2 b_i \left[t\frac{\partial}{\partial x_i} + \frac{\partial}{\partial V_i} - \frac{1}{c} e_{isk} B_s \frac{\partial}{\partial E_k} + \rho\frac{\partial}{\partial j_i} \right]. \tag{9}$$

The distribution functions of the plasma particles are invariant under subgroup (6), and they are subject to group transformations in accordance with (7) and (8). The parameters of the theory are invariants of subgroup (6), (7). The infinitesimal operators in (8) are the basis for the construction of a renormalization group. In particular, in this renormalization-group context,[8] the Lorentz group with the infinitesimal operators Y_i, Z_i has been used to reproduce the nonlinear dielectric constants of a homogeneous hot plasma without external fields from the corresponding expressions in a cold plasma at rest. The infinitesimal form of group (6)–(8) is characteristic in group analysis.[2–4] Finite transformations of the group are constructed by solving the Lie equations for the infinitesimal operators. For example, the one-dimensional Lorentz transformations[9] are solutions of the Lie equations for the x component of the vector infinitesimal operator Y_i. Multidimensional Lorentz transformations, along with circular rotations,[10] are the solutions of the Lie equations for six scalar infinitesimal operators corresponding to the two vector operators Y_i and Z_i in Poincaré subgroup (6).

The first calculation of a continuous Lie point group allowed by a Vlasov kinetic equation was carried out in Ref. 11 in the one-dimensional nonrelativistic approximation. It was carried out for a homogeneous electron plasma with a self-consistent potential electric field, on the basis of a reformulation of the integrodifferential relations as an equivalent infinite system of differential equations for power moments of the electron velocity distribution. A decade and a half later, the result of Ref. 11 was reproduced in Ref. 5 on the basis of a *unified* general approach developed in that paper. That approach makes it possible to construct and solve the governing equations of a group and to directly take account of the nonlocal (global) contributions in the initial equations subject to the group analysis. The approach of Ref. 5 has made it possible for us to derive group (6)–(8) for the system of integrodifferential equations in (1)–(3), which is qualitatively different from that in Ref. 11. Since the operation of integrating the distribution functions f and $\bar{f}$ over the momenta $\vec{p}$ and $\vec{q}$ on the right sides of (3) is nonlocal, the derivatives with respect to f and $\bar{f}$ in (7), (8) are to be understood in the Fréchet sense in the construction of the governing equations of the group. The nonlocal governing equations can be split up in this case through a simple variational differentiation with respect to the independent group variables f and $\bar{f}$.

We note in conclusion that the result in (6)–(8) remains valid for the exact microscopic equations of Klimontovich[12] if the one-particle distribution functions $f, \bar{f}$ and the average electric and magnetic fields in (6)–(8) are replaced by the microscopic phase densities N, $\bar{N}$ and the microscopic fields $\vec{E},\vec{B}$. The latter incorporate fluctuations of the plasma. An additional expansion of group (6)–(8) can be achieved by (for example) a Lagrange replacement of the Euler momenta $\vec{p}$ and $\vec{q}$ in kinetic equations (1).

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