Solutions of the Yang–Mills equation in $d=4n$ dimensions for an arbitrary gauge group

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The Yang–Mills equations in the space $\mathbb{R}^{4n}$ are reduced to a system of equations which break up into the Nahm equations and linear equations for a scalar field φ. This is done for gauge fields of an arbitrary semisimple Lie group G.

1. Solutions of the Yang–Mills equations in the Euclidean spaces $\mathbb{R}^4$ and $\mathbb{R}^8$ were recently used to derive soliton solutions in heterotic-strong theory.[1] Finding Yang–Mills solutions in $\mathbb{R}^{4n}$ is thus important for learning about nonperturbative effects in string theory.

In the Euclidean space $\mathbb{R}^{4n}$, with metric δ_{ab}, we consider gauge fields A_a of the semisimple Lie group G, with the field $F_{ab}=\partial_a A_b-\partial_b A_a+[A_a,A_b]$, $a,b,...=1,...,4n$, $n=1,2,...$. The Yang–Mills equations for A_a are

$$\partial_a F_{ab}+[A_a,F_{ab}]=0. \tag{1}$$

Equations (1) in $\mathbb{R}^{4n}$ were anlayzed in Refs. 2 and 3. It was shown in those papers that the Corrigan–Fairlie–'t Hooft–Wilczek ansatz[4] for gauge fields of the SU(2) group can be generalized to a dimensionality $d=4n$. In the present letter we generalize this ansatz to gauge fields of an *arbitrary* semisimple Lie group G. We describe corresponding new classes of solutions of the Yang–Mills equations in the space $\mathbb{R}^{4n}$.

2. In $\mathbb{R}^{4n}$ one can always specify three constant antisymmetric tensors J^1_{ab}, J^2_{ab}, and J^3_{ab}, whose components satisfy the relations[5]

$$J^{\alpha}_{ac}J^{\beta}_{bc}=\delta^{\alpha\beta}\delta_{ab}+\epsilon^{\alpha\beta\gamma}J^{\gamma}_{ab}. \tag{2}$$

Here $\epsilon_{\alpha\beta\gamma}$ are structure constants of the $SU(2)$ group, and $\alpha,\beta,...=1,2,3$.

For A_a we consider the ansatz

$$A_a=-J^{\alpha}_{ac}T_{\alpha}(\varphi)\partial_c\varphi, \tag{3}$$

where the constants J^{α}_{ac} satisfy (2), φ is an arbitrary function of the coordinates $x^a\in|\mathbb{R}^{4n}$, and $\partial_c:=\partial/\partial x^c$. The functions T_α depend on φ and take on values in the Lie algebra $\mathscr{G}$ of the gauge group G; i.e., these are matrix functions. Ansatz (3) generalizes the Corrigan–Fairlie–'t Hooft–Wilczek ansatz[4] to a dimensionality $d=4n$ and to an arbitrary group G. In $d=4$, ansatz (3) becomes the ansatz of Refs. 6 and 7.

Substituting (3) into the definition of F_{ab}, we find

$$F_{ab}=J^{\alpha}_{ac}\{T_\alpha\partial_b\partial_c\varphi+\dot T_\alpha\partial_b\varphi\partial_c\varphi\}-J^{\alpha}_{bc}\{T_\alpha\partial_a\partial_c\varphi+\dot T\partial_a\varphi\partial_c\varphi\}$$
$$+J^{\alpha}_{ac}J^{\beta}_{be}[T_\alpha,T_\beta]\partial_c\varphi\,\partial_e\varphi, \tag{4}$$

where $\dot T_\alpha:=dT_\alpha/d\varphi$. Using (2), we can easily show that

$$\partial_aF_{ab}+[A_a,F_{ab}]=T_\alpha j^{\alpha}_{ab}\partial_a(\Box\varphi)+[\dot T_\alpha,T_\alpha]\partial_c\varphi\,\partial_c\varphi\,\partial_b\varphi\cdot$$
$$(\ddot T_\alpha-[T_\beta,[T_\alpha,T_\beta]])J^{\alpha}_{ab}\,\partial_a\varphi\,\partial_c\varphi\,\partial_c\varphi+\epsilon_{\alpha\beta\gamma}(\epsilon_{\beta\gamma\delta}\dot T_\delta$$
$$+[T_\beta,T_\gamma])J^{\alpha}_{ca}\,\partial_a\varphi\,\partial_c\partial_b\varphi-2(\epsilon_{\alpha\beta\gamma}\dot T_\gamma+[T_\alpha,T_\beta])J^{\alpha}_{ac}J^{\beta}_{be}\,\partial_a\varphi\,\partial_c\partial_e\varphi$$
$$+\dot T_\alpha\partial_a\varphi\{2J^{\alpha}_{ac}\,\partial_c\partial_b\varphi-2J^{\alpha}_{bc}\,\partial_c\partial_a\varphi+2\epsilon^{\alpha}_{\beta\gamma}J^{\beta}_{ac}J^{\gamma}_{be}\,\partial_c\partial_e\varphi+J^{\alpha}_{ab}\,\Box\varphi\}, \tag{5}$$

where $\Box:=\partial_c\partial_c$.

Yang-Mills equations (1) hold if the following equations are satisfied:

$$2J^{\alpha}_{ac}\,\partial_c\partial_b\varphi-2J^{\alpha}_{bc}\,\partial_c\partial_a\varphi+2\epsilon^{\alpha}_{\beta\gamma}\,J^{\beta}_{ac}J^{\gamma}_{be}\,\partial_c\partial_e\varphi+J^{\alpha}_{ab}\,\Box\varphi=0, \tag{6a}$$

$$\epsilon_{\alpha\beta\gamma}\dot T_\gamma+[T_\alpha,T_\beta]=0. \tag{6b}$$

To show this, we differentiate Eqs. (6a) with respect to x^a. We find $J^{\alpha}_{bc}\partial_c(\Box\varphi)=0$. It is easy to see that we have $[\dot T_\alpha,T_\alpha]=0$ by virtue of (6b) and the Jacobi identity for the matrices T_α. If we then differentiate (6b) with respect to φ and again use (6b), we find $\ddot T_\alpha-[T_\beta,[T_\alpha,T_\beta]]=0$. Consequently, if Eqs. (6) hold, then the right side of (5) vanishes, and the Yang–Mills equations are satisfied.

3. To find solutions of Eqs. (6a), we replace the indices a, b,... by the double indices $(\mu i),(\nu i)$,..., where $\mu,\nu,...=1,...,4$, $i,j,...=1,...,n$. The tensors $J^{\alpha}_{(\mu i)(\nu j)}\ (=J^{\alpha}_{ab})$ can be chosen in the form

$$J^{\alpha}_{(\mu i)(\nu j)}=\delta_{ij}\eta^{\alpha}_{\mu\nu}, \tag{7}$$

where $\eta^{\alpha}_{\mu\nu}$ are the 't Hooft tensors. By definition (see, for example, Ref. 8), we have $\eta^{\alpha}_{\beta\gamma}=\epsilon^{\alpha}_{\beta\gamma}$, $\eta^{\alpha}_{\mu 4}=-\eta^{\alpha}_{4\mu}=\delta^{\alpha}_{\mu}$, $\alpha,\beta,\gamma=1,2,3$. The tensors $\eta^{\alpha}_{\mu\nu}$ satisfy the identities[8]

$$\eta^{\alpha}_{\mu\lambda}\,\eta^{\beta}_{\nu\lambda}=\delta^{\alpha\beta}\,\delta_{\mu\nu}+\epsilon^{\alpha\beta\gamma}\,\eta^{\gamma}_{\mu\nu}, \tag{8a}$$

$$\epsilon^{\alpha}_{\beta\gamma}\eta^{\beta}_{\mu\lambda}\eta^{\gamma}_{\nu\sigma}=\delta_{\mu\nu}\,\eta^{\alpha}_{\lambda\sigma}-\delta_{\mu\sigma}\,\eta^{\alpha}_{\lambda\nu}-\delta_{\lambda\nu}\,\eta^{\alpha}_{\mu\sigma}+\delta_{\lambda\sigma}\,\eta^{\alpha}_{\mu\nu}. \tag{8b}$$

It is easy to verify that tensors (7) satisfy (2) by virtue of identities (8a).

Substituting (7) into Eqs. (6a), and using identities (8), we find

$$2\eta^{\alpha}_{\mu\lambda}(\partial_{\lambda i}\partial_{\nu j}\varphi-\partial_{\lambda j}\partial_{\nu i}\varphi)-2\eta^{\alpha}_{\nu\lambda}(\partial_{\lambda j}\partial_{\mu i}\varphi-\partial_{\lambda i}\partial_{\mu j}\varphi)$$
$$+\delta_{\mu\nu}\eta^{\alpha}_{\lambda\sigma}(\partial_{\lambda i}\partial_{\sigma j}\varphi-\partial_{\lambda j}\partial_{\sigma i}\varphi)+\eta^{\alpha}_{\mu\nu}(2\partial_{\lambda i}\partial_{\lambda j}\varphi+\delta_{ij}\Box\varphi)=0, \tag{9}$$

where $\partial_{\lambda i}:=\partial/\partial x^{\lambda i}$. It is easy to see that Eqs. (9) are equivalent to the equations

$$\partial_{\mu i}\partial_{\nu j}\varphi=\partial_{\mu j}\partial_{\nu i}\varphi,\quad \partial_{\lambda i}\partial_{\lambda j}\varphi=0, \tag{10}$$

where μ and ν take on arbitrary values from 1 to 4, and i and j take on arbitrary values from 1 to n. From (10) we find $\Box\varphi = \Sigma_{j=1}^{n}\partial_{\lambda j}\partial_{\lambda j}\varphi = 0$.

4. Equations (10) arise during the construction of metrics on hyper-Kähler manifolds of dimensionality $4n$ (Ref. 5). A general solution of Eqs. (10) can be written as a contour integral of a holomorphic function in the auxiliary complex variable ζ (Ref. 5). In particular, for the space $\mathbb{R}^8$ an explicit expression for the general solution φ in terms of a contour integral was written by Ward.[2] In this form, however, the solution is of little use for applications. We will accordingly write three fairly general solutions in explicit form.

Example 1. The simplest solution is the function

$$\varphi = p_a x^a = p_{\mu i} x^{\mu i}, \tag{11}$$

where p_a is a constant vector in $\mathbb{R}^{4n}$.

Example 2. By definition we set $X_\mu = x_{\mu i} p_i$, $p_i = \text{const}$. We assume that the function φ depends on X_μ alone; i.e., we assume $\varphi = \varphi(X_1, X_2, X_3, X_4)$. It is easy to show that in this case Eqs. (10) reduce to a Laplace equation in terms of the "collective" coordinates X_μ: $\partial^2\varphi/\partial X_\mu \partial X_\mu = 0$. As a solution we could choose, for example,

$$\varphi = 1 + \sum_{I=1}^{q} \frac{B_I^2}{(X_\mu - C_\mu^I)(X_\mu - C_\mu^I)}, \tag{12}$$

where B_I, $C_\mu^I = \text{const}$, and q is any natural number.

Example 3. We assume that the function φ_i depends on only the coordinates $x_{\mu i}$ with index i: $\varphi_i = \varphi_i(x_{1i}, x_{2i}, x_{3i}, x_{4i})$. We set

$$\varphi = \sum_{i=1}^{n} \varphi_i. \tag{13}$$

We substitute (13) into (10). It is easy to see that Eqs. (10) reduce to Laplace equations for φ_i in terms of the coordinates $x_{\mu i}$: $\partial_{\lambda i}\partial_{\lambda i}\varphi_i = 0$, where no summation over the index i is intended. Consequently, by taking any n functions φ_i which satisfy the Laplace equations, we obtain solution (13) of Eqs. (10) [and (6a)].

We wish to stress that the gauge fields which correspond to (13) *are not* the direct sum of n solutions in four-dimensional subspaces, since the $T_\alpha(\varphi)$ depend in a nontrivial way on φ and cannot be decomposed into the direct sum of n matrices.

5. Equations (6b) are called the "Nahm equations."[9,10] They arise in the construction of solutions of the Yang–Mills equations in $\mathbb{R}^4$ (Refs. 11, 12, 6, 7) and in the chiral-field model in $\mathbb{R}^2$ (Ref. 13). The Nahm equations have a Lax representation with a spectral parameter.[9,10,13] In terms of theta functions, we can write a general solution of Eqs. (6b) for any semisimple Lie algebra $\mathcal{G}$ (see the discussion in Refs. 9, 10, and 13). A particular class of solutions of these equations can be found through a reduction to equations of a Toda chain. Explicit expressions for the various ansatzes for $T_\alpha(\varphi)$, solutions, and a discussion thereof can be found in Refs. 10, 6, 7, and 12. Consequently, any solution of system of equations (6) gives us solution (3) of the Yang–Mills equations in $\mathbb{R}^{4n}$.

Ansatz (3) is associated with the existence in the $\mathbb{R}^{4n}$ of three tensors J^{α}_{ab} (a *quarternionic structure*). Corresponding ansatzes associated with an *octonionic structure* can be used to find other solutions of the Yang–Mills equations in $\mathbb{R}^7$ and $\mathbb{R}^8$ (Ref. 14).

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