Exactly solvable multichain quantum model

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Exact results are derived for a new multi-index magnetic system by the method of the quantum inverse scattering problem. Solutions of the Yang–Baxter equation are found. The expression for the ground-state energy is analyzed.

The discovery of the metal oxide superconductors[1] has greatly heightened interest in 2D quantum systems. Most such systems are studied theoretically by some modification or other of the mean-field method, by variational methods, or by other approximate methods.[2] In 2D systems, however, approximate methods may lead to results which are incorrect even in the qualitative sense. There is accordingly particular interest in any 2D quantum models, even those which are a bit detached from reality. Most of the exact solutions of 2D models of statistical mechanics and 1D quantum models have been found through the construction of commuting transfer matrices. There are several 2D quantum models (Refs. 3 and 4, for example), for which commuting transfer matrices have been constructed. In this letter we propose an exactly solvable 2D spin quantum model in which the interaction between nearest chains in a plane is an interaction between sites lying within the first coordination sphere. We write expressions for the transfer matrix and the ground state energy. We also write equations for an algebraic Bethe ansatz.

We consider the behavior of N spin chains ($s = 1/2$) with the Hamiltonian

$$H = -(1/2)\sum_{n=1}^{N}\sum_{m=1}^{M}\{(\sigma_{n,m}^{+}\sigma_{n,m+1}^{-}\exp(4i\phi(\sigma_{n-1,m}^{z} - \sigma_{n+1,m}^{z})) + \text{H.c.}) + 2\Delta^{(n)}\sigma_{n,m}^{z}\sigma_{n,m+1}^{z} + 2h\sigma_{n,m}^{z}\}, \quad (1)$$

where $\Delta^{(n)}$ are the exchange-anisotropy constants for each chain; $\sigma_{n,m}^{\pm} = \sigma_{n,m}^{x} \pm i\sigma_{n,m}^{y}$;

$\sigma^x_{n,m}$, $\sigma^y_{n,m}$, $\sigma^z_{n,m}$ are the x, y, z spin projections for site m of chain n, respectively; ϕ is the constant of the interaction between spin chains in the plane; and h is the magnetic field.

The Jordan–Wigner transformations,[5] modified for the multi-index case, relate the spin system to the system of fermions, whose Hamiltonian is

$$H = -(1/2)\sum_{n=1}^{N}\sum_{m=1}^{M}\{(a^+_{n,m+1}a_{n,m}\exp(4i\phi(a^+_{n+1,m}a_{n+1,m} - a^+_{n-1,m}a_{n-1,m}))$$

$$\text{H.c.}) + 2\Delta^{(n)}(1 - 2a^+_{n,m}a_{n,m})(1 - 2a^+_{n,m+1}a_{n,m+1}) + 2h(1 - 2a^+_{n,m}a_{n,m})\}. \quad (2)$$

An exact solution of a 2D system of interacting spin-zero fermions can also be found in this manner.

For a chain with "free ends" there exists a unitary gauge transformation which reduces Hamiltonian (1) to the Hamiltonian of N noninteracting spin chains. No such unitary transformation exists in the case of periodic boundary conditions. In our model the interaction between spin chains is thus manifested primarily in corrections of finite size. This interaction leads to effects analogous to the Aharonov–Bohm effect and is of a quantum nature.[6] Corrections of finite size are particularly important for mesoscopic systems. These corrections have recently been studied in several theoretical papers by the method of the quantum inverse problem.[7,8]

The entity of primary interest in the quantum method of the inverse scattering problem is a local L operator. For our case of N spin chains, this operator is

$$\mathbb{L}_m(\lambda) = \overrightarrow{\prod_{n=1}^{N}}\tilde{L}^m_n(\lambda^{(n)})$$

$$\tilde{L}^m_n(\lambda^{(n)}) = \exp(i\phi\sigma^z_{O,m}\sigma^z_{n+1,m})L^m_n(\lambda^{(n)})\exp(-i\phi\sigma^z_{O,m}\sigma^z_{n-1,m});$$

$$L^m_n(\lambda^{(n)}) = w_1(\lambda^{(n)})\sigma^x_{O,n}\sigma^x_{m,n} + w_2(\lambda^{(n)})\sigma^y_{O,n}\sigma^y_{m,n}$$

$$+w_3(\lambda^{(n)})\sigma^z_{O,n}\sigma^z_{m,n} + w_4(\lambda^{(n)});$$

where m is the index of the site, n is the index of the chain, and O represents an auxiliary subspace. The functions $w_k(\lambda)$ constitute the solution of the Yang-Baxter equation[10]

$$\mathbb{R}(\lambda-\mu)(\mathbb{L}(\lambda)\otimes\mathbb{L}(\mu)) = (\mathbb{L}(\mu)\otimes\mathbb{L}(\lambda))\mathbb{R}(\lambda-\mu);$$

$$\mathbb{R}(\lambda) = \overrightarrow{\prod_{n=1}^{N}}P_{O,n;m,n}\mathbb{L}(\lambda),$$

where $P_{i,k}$ is the permutation operator in subspaces i and k,

$$w_1(\lambda) = w_2(\lambda) = \sin(2\eta)/(2\sin(\lambda+\eta));$$

$$w_{3,4}(\lambda) = (1 \pm \sin(\lambda-\eta))/(2\sin(\lambda+\eta)).$$

The Yang–Baxter equation is a necessary and sufficient condition for the commutativity of transfer matrices with nonidentical arguments. In our case the transfer matrix is

$$\mathbf{T}(\lambda) = \mathrm{Trace} \overbrace{\prod_{n=1}^{N}} \overbrace{\prod_{m=1}^{M}} \tilde{L}_n^m(\lambda). \tag{3}$$

The trace in (3) is over the auxiliary subspace; M is the number of sites. This commutativity means that we can find the complete set of conserved operators which commute with the transfer matrix $T(\lambda)$. In particular, Hamiltonian (1) is the logarithmic derivative with $\lambda^{(n)} = \eta^{(n)}$, where $\cos(2\eta^{(n)}) = \Delta^{(n)}$.

Going through the standard procedure,[9] we find the eigenvalues of the transfer matrix G:

$$G = \prod_{n=1}^{N} \exp[2i\phi(\nu_{n-1} - \nu_{n+1})] \prod_{k_n=1}^{\nu_n} \frac{\sin(\lambda_{k_n}^{(n)} - \lambda^{(n)} + \eta^{(n)})}{\sin(\lambda_{k_n}^{(n)} - \lambda^{(n)} - \eta^{(n)})}$$

$$+ \exp[-2i\phi(\nu_{n-1} - \nu_{n+1})] \frac{\sin^M(\lambda^{(n)} + \eta^{(n)})}{\sin^M(\lambda^{(n)} - \eta^{(n)})} \prod_{k_n=1}^{\nu_n} \frac{\sin(\lambda^{(n)} - \lambda_{k_n}^{(n)} + \eta^{(n)})}{\sin(\lambda^{(n)} - \lambda_{k_n}^{(n)} - \eta^{(n)})}.$$

The quantum numbers $\lambda_{k_n}^{(n)}$ are determined from the system of equations

$$\exp[4i\phi(\nu_{n-1} - \nu_{n+1})] \left(\frac{\sin(\lambda_{j_n}^{(n)} + \eta^{(n)})}{\sin(\lambda_{j_n}^{(n)} - \eta^{(n)})}\right)^M = \prod_{\substack{i_n=1 \\ i_n \neq j_n}}^{\nu_n} \frac{\sin(\lambda_{j_n}^{(n)} - \lambda_{i_n}^{(n)} + 2\eta^{(n)})}{\sin(\lambda_{j_n}^{(n)} - \lambda_{i_n}^{(n)} - 2\eta^{(n)})}. \tag{4}$$

The quantity ν_n in (4) is the number of down spins in chain n; $\nu_{N+1} = \nu_O = M/2$. Note that the index n indicates that the constants of the intrachain interaction for the different sublattices are generally not the same.

It is a simple matter to find the eigenvalue of the state in which there are ν_n down spins in chain n for Hamiltonian (1):

$$E = \sum_{n=1}^{N} \left\{ -\frac{M}{2}(\Delta^{(n)} + h) + h\nu_n \right.$$

$$\left. + \sum_{j=1}^{\nu_n} \left[2\Delta^{(n)} - \frac{\sin(\lambda_{j_n}^{(n)} + \eta^{(n)})}{\sin(\lambda_{j_n}^{(n)} - \eta^{(n)})} - \frac{\sin(\lambda_{j_n}^{(n)} - \eta^{(n)})}{\sin(\lambda_{j_n}^{(n)} + \eta^{(n)})} \right] \right\}. \tag{5}$$

To find the ground-state energy E_O we need to find the quantum numbers $\lambda_{k_n}^{(n)}$ from (4) for the given ν_n, substitute them into expression (5), and minimize E_O with

respect to ν_n. So far, no way has been found to carry out this procedure analytically for the general case. In certain limiting cases, in contrast, we have found an expression for the ground-state energy.

For $\Delta^{(n)} = 0$, for example, which corresponds to free fermions, the ground-state energy is

$$E = -\sum_{n=1}^{N} \left\{ \frac{2\sin(\pi\nu_n/M)\cos[(4\phi(\nu_{n-1} - \nu_{n+1}) + \pi)/M]}{\sin(\pi/M)} - 2h\nu_n \right\}. \tag{6}$$

It can be seen from expression (6) that the leading term in the expansion of the energy in $1/M$ does not depend on the constant (ϕ) of the interaction between sublattices in the absence of a magnetic field. As expected, this interaction makes a contribution on the order of $1/M$. For $N = 2$ and $h = 0$, for example, we find

$$E_O \approx -(4M/\pi) + (2/M)\{(\pi/3) + (2\phi)^2[1 - (2\pi)^2]/\pi[1 + (4\phi)^2]\}.$$

For nonzero h, the situation is more interesting, since a dependence on ϕ may be contained even in the leading term in $1/M$. For the case of two chains, for example, the ground-state energy is

$$E_O \approx -2M\{[2\sin(\pi x)\cos(4\phi(x - 1/2))/\pi] - 2hx\},$$

where z is found from the transcendental equation

$$\cos(\pi x)\cos(4\phi(x - 1/2)) - 4\phi\sin(\pi x)\sin(4\phi(x - 1/2)) = h.$$

As $\Delta^{(n)} \to \infty$, the system obviously degenerates into N independent Ising subsystems, and the energy of the ground state is trivial in this case.

For $-1 < \Delta^{(n)} < 1$, we can use the results of Ref. 8 for the correction on the order of $1/M$ under "twisted" boundary conditions:

$$E = \sum_{n=1}^{N} \left\{ M\epsilon_\infty^n - \frac{\pi v_F^{(n)}}{6M} \left[1 - \frac{3(\nu_n - \mu_n(h)M)^2}{\zeta_n^2(\Lambda)} \right.\right.$$

$$\left.\left. - 3\zeta_n^2(\Lambda)(4\phi/\pi)^2(\nu_{n-1} - \nu_{n+1})^2 \right] \right\}.$$

Here ϵ_∞^n is the ground-state energy for each chain in the case $\nu_n = M/2$ (Ref. 11); we have $\mu_n(h) = \nu_n/M$ in the limit $M \to \infty$; $v_F^{(n)}$ is the Fermi velocity; and the quantities $\zeta_n(\Lambda)$ are the "dressed" charges, calculated by analogy with Ref. 8. Note that in the case of toroidal boundary conditions, $\nu_{N+1} = \nu_1$, the behavior of the system is trivial: We have $\nu_n = M/2$ for all n.

Wang[12] has recently proposed a generalization of the Jordan–Wigner transformations for a 2D quantum spin system. The nature of the interaction between Fermi particles in neighboring chains in Hamiltonian (2) of the present paper is similar to Hamiltonian (3) of Ref. 12, since in each case the interaction makes the phase-factor

exponential functions dependent on the occupation numbers of the sites in neighboring chains (i.e., on the site magnetizations of the sublattices).

In summary, a new class of exactly solvable quantum models, which are effectively two-dimensional, has been analyzed in this paper by the method of the quantum inverse scattering problem. These models are infinite sets of spin chains with a nontrivial interaction between nearest chains. Each of these chains is exactly integrable in the absence of an interchain interaction. In a zero magnetic field, the interaction between chains makes a contribution on the order of $1/M$ to the energy. A nonzero magnetic field may cause the leading term in the expression for the ground-state energy to acquire a substantial dependence on the interchain–interaction constant.

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