Quantum Hall state and chiral edge state in thin ^{3}He-A film

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The chiral gapless excitations at the boundary of the superfluid ^{3}He-A film give rise to quantization of the transverse (Hall) conductivity in the absence of a magnetic field.

The edge states of fermions have been recently discussed for a droplet of two-dimensional electron gas which exhibits the quantum Hall effect (QHE) under applied magnetic field.[1–3] The fermions at the boundary of the droplet are chiral and gapless and thus represent the only low-energy fermionic excitations in this system, since there is a finite energy gap for fermions within the droplet. The neutral superfluid ^{3}He-A film represents the QHE without a magnetic field:[4] The response of the particle current to the gradient of the chemical potential applied in the transverse direction (the analog of Hall conductivity) exhibits quantization. However, as was shown in Ref. 4, this quantization rule is only approximate and is valid in the extreme limit of the small gap as compared with the Fermi energy. Here we show that in the geometry which leads to the existence of the chiral edge excitations in the ^{3}He-A film the quantization of the Hall conductivity is exact.

For the neutral ^{3}He-A film the role of the magnetic field is played by the spontaneous orbital angular momentum of the Cooper pairs, which violates the time inversion and 2D space inversion symmetries. The direction of the momentum, denoted by unit vector $\hat{l}$, is fixed along the normal to the film: $\hat{l} = \pm\hat{z}$. We consider here the QHE in the following geometry (see Fig. 1): the difference of chemical potentials $\mu(x_2) - \mu(x_1)$ is applied to the strip $x_1 < x < x_2$ of the film with the given orientation $\hat{l} = \hat{z}$ and the mass current J is measured in the direction y. We consider three different geometries: (1) outside the layer there is no ^{3}He-A; instead, for example, there is the planar state which has no orbital momentum (Fig. 1a); (2) the ^{3}He-A film with an opposite orientation of $\hat{l} = -\hat{z}$ (Fig. 1b) is outside the strip; (3) the ^{3}He-A film everywhere has the same orientation of $\hat{l}$ (Fig. 1c). Our results for the total current in the strip for these cases are

$$J^{(1)} = \frac{N}{2}\frac{m_3}{2\pi\hbar}(\mu(x_2) - \mu(x_1)) \quad , \tag{1}$$

$$J^{(2)} = N\frac{m_3}{2\pi\hbar}(\mu(x_2) - \mu(x_1)) - \frac{\hbar}{4}(\rho(x_2) - \rho(x_1)) \quad , \tag{2}$$

$$J^{(3)} = \frac{\hbar}{4}(\rho(x_2) - \rho(x_1)) \quad , \tag{3}$$

where ρ is the particle density per unit area of the film, m_3 is the mass of ^{3}He atom,

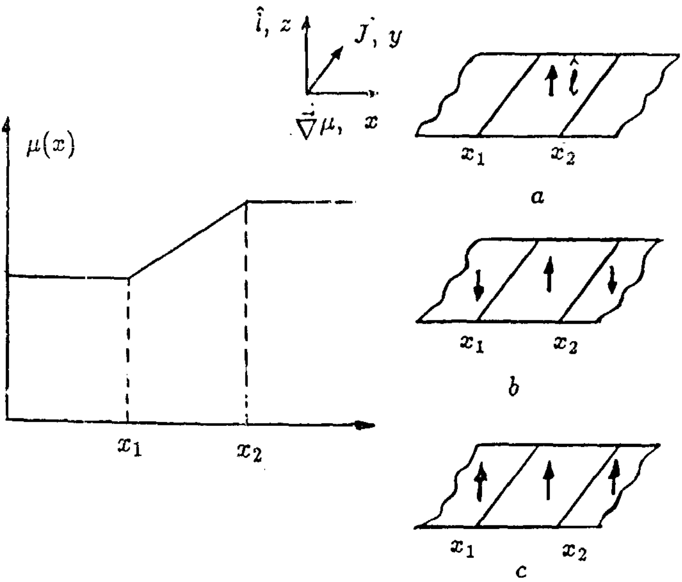

FIG. 1. Profile of the chemical potential applied to the strip of the ^{3}He-A film in three different geometries. (a) There is no $\hat{l}$ vector outside the layer; (b) $\hat{l}$ is oriented in the opposite direction outside the layer; (c) $\hat{l}$ is oriented everywhere in the same direction.

and N is related to the number of the chiral fermions at the boundary of the layer, which depends on the film thickness and increases with increasing thickness. N is even for the ^{3}He-A film and is an integer for the ^{3}He-A_1 film. Since this result does not depend on the particular features of the system, we calculate the current using the simplest model for the ^{3}He-A film.

In the thin ^{3}He film the dimensional quantization along z is important, and the quasiparticle spectrum in the normal ^{3}He film depends on the index q of discrete level of the motion along z and on the two-dimensional momentum $\vec{k}=(k_x,k_y)$. For the simplest model of the noninteracting levels q the spectrum of the particles at each level of a normal film is

$$\varepsilon_q(\vec{k})=\varepsilon_q(0)+\frac{k^2}{2m_3}\quad , \tag{4}$$

where $\varepsilon_q(0)\sim\hbar^2q^2/m_3a^2$; here a is the thickness of the film. At each level q, which is below the chemical potential, $\varepsilon_q(0)<\mu$, the Fermi liquid is formed with its own Fermi momentum k_{Fq}:

$$\frac{k_{Fq}^2}{2m_3}=\mu-\varepsilon_q(0)\quad . \tag{5}$$

The Cooper pairing leads to nondiagonal matrix elements between the particle and hole states. The relevant Bogolyubov–Nambu matrix for the fermions at the q

level in the ^{3}He-A film is the 2×2 matrix, provided that the spin part of the order parameter is discarded and the interlevel interaction is ignored. It is expressed in terms of two vectors $\hat{e}^1$ and $\hat{e}^2$ in the x,y plane:

$$\mathrm{H}_q = (\varepsilon_q(\vec{k}) - \mu)\tau_3 + c_q\vec{k}\cdot\hat{e}^1\ \tau_1 - c_q\vec{k}\cdot\hat{e}^2\ \tau_2 \quad , \tag{6}$$

where $\vec{\tau}$ are the Pauli matrices in the particle-hole space, and c_q are the amplitudes of the nondiagonal elements. In the equilibrium ^{3}He-A film $\hat{e}^1 \perp \hat{e}^2$, $|\hat{e}^1| = |\hat{e}^2| = |1$, and $\hat{l} = \hat{e}^1\times\hat{e}^2$. The energy spectrum

$$E_q^2(\vec{k}) = (\varepsilon_q(\vec{k}) - \mu)^2 + c_q^2[(\vec{k}\cdot\hat{e}^1)^2 + (\vec{k}\cdot\hat{e}^2)^2] \tag{7}$$

is nowhere zero in the equilibrium ^{3}He-A film, by analogy with the two-dimensional electron gas in the QHE regime.

The zeros in the spectrum appear at $x = x_1$ and at $x = x_2$, i.e., at the edges of the ^{3}He-A strip. We consider first the case (2), where the edges of the layer are the borders between the domains with different $\hat{l}$ orientations. Since the topological properties of the spectrum are insensitive to the details, we chose the simplest realization[5] for $\hat{e}^1(x)$ and $\hat{e}^2(x)$:

$$\hat{e}^1(x) = \hat{x} \quad , \quad \hat{e}^2(x) = -\hat{y}\ \tanh\left(\frac{x - x_1}{\xi_B}\right)\tanh\left(\frac{x - x_2}{\xi_B}\right) \quad , \tag{8}$$

where ξ_B is the size of the domain boundary, which is on the order of the coherence length, $\xi_B \gg K_F^{-1}$. We assume that $\xi_B \ll (x_2 - x_1)$. Far from the boundaries $\hat{e}^1 = -\hat{y}$ at $x < x_1$ and $x > x_2$, and $\hat{e}^2 = \hat{y}$ at $x_1 < x < x_2$, which corresponds to $\hat{l} = -\hat{z}$ at $x < x_1$ and $x > x_2$, and $\hat{l} = \hat{z}$ at $x_1 < x < x_2$.

Since $\xi_B \gg k_F^{-1}$, we may first consider the spectrum in the semiclassical approximation, in which the spectrum depends on the momentum and on the coordinate: $E(\vec{k}, \vec{r})$ in Eq. (7) with $\hat{e}^1(x)$ and $\hat{e}^2(x)$ from Eq. (8). The energy is zero at the lines ($x = x_1$, $k = 0$, $k_y = \pm k_{Fq}$) and ($x = x_2$, $k_x = 0$, $k_y = \pm k_{Fq}$) in the 4-dimensional ($\vec{k}, \vec{r}$) space. These are the straight lines along the y axis of 4D space. These manifolds of zeros have the topological stability. The explicit expression for the topological invariant, which supports the stability of the zeros, may be constructed[4] in terms of the Green's function matrix $G_{qp}(\omega, \vec{k}, x)$:

$$m = \frac{1}{24\pi^2} e_{\mu\nu\lambda\gamma}\ \mathrm{tr} \int dS^\gamma\ \ \mathrm{G}\partial_\mu \mathrm{G}^{-1}\mathrm{G}\partial_\nu \mathrm{G}^{-1}\mathrm{G}\partial_\lambda \mathrm{G}^{-1} \quad . \tag{9}$$

For the given value of y the integral is taken over the 3D sphere in 4D space (ω, k_x, k_y, x) about each zero point of the spectrum, say, ($\omega = 0$, $x = x_1$, $k_x = 0$, $k_y = k_{Fq}$). This integral is $m = 1$ for zeros at $x = x_1$ and $m = -1$ for zeros at $x = x_2$, which can be checked in the model of the noninteracting levels, where the Green's function matrix is diagonal in level q indices:

$$\mathrm{G}_{qp}(\omega, \vec{k}, x) = \delta_{qp}\frac{1}{i\omega + \mathrm{H}_q(\vec{k}, x)} \quad . \tag{10}$$

The number of zeros in the semiclassical energy spectrum $E_q(\vec{k}, \vec{r})$ is thus $4q_0$ for each domain boundary, where $2q_0$ is the number of the Fermi liquids in the normal state: q_0 is the number of the levels of the quantized motion along z below μ, and we take into account the double degeneracy over spin; for the ^{3}He-A_1, where only one spin component forms the Cooper pairs, there is no factor 2.

The index theorem relates the number of zeros in the semiclassical spectrum with the same number $4q_0$ of the gapless fermionic modes which are localized at the boundary in the exact quantum-mechanical problem. The exact energy spectum of the fermions, $E_n(k_y)$, depends on the momentum k_y along the domain boundary. In the simplest realization of the structure of the domain boundary,[5] the Hamiltonian which defines the spectrum, say, at $x=x_1$ and $k_y \approx k_{Fq}$ is

$$\mathrm{H}_q = v_{Fq}(k_y - k_{Fq})\tau_3 + c_q\tau_1(-i\frac{\partial}{\partial x}) - c_q k_{Fq}\tanh\frac{x-x_1}{\xi_B}\ \tau_2 \quad . \tag{11}$$

Each Hamiltonian has zero-mode eigenfunction, the spinor $\Psi=(u(x),\ v(x))$:

$$\Psi = (\ 0\ ,\ \cosh^{-s}\frac{x}{\xi_B},)\quad ,\quad s = k_{Fq}\xi_B \quad . \tag{12}$$

Each mode produces the gapless branch of the fermionic spectrum, which crosses zero value at $k_y = k_{Fq}$. These are one-dimensional Fermi liquids.

It is important that the symmetry with respect to $k_y \to -k_y$ is broken here: in the vicinity of the Fermi points $\pm k_{Fq}$ the spectrum is

$$E_0(q,k_y) = \mathrm{sign}(k_y)\ \ (\varepsilon_q(k_y) - \mu) \approx v_{Fq}(k_y \mp k_{Fq}) \quad , \tag{13}$$

which corresponds to the right moving zero fermionic modes at the domain boundary. There are only $4q_0$ right moving gapless fermions localized at the boundary at $x=x_1$ and the same number of the left moving chiral gapless fermions are localized at $x=x_2$.

Because of this asymmetry, there is net linear momentum and therefore a ground-state mass current in each of the domain boundaries. The magnitude of the vacuum current can be determined by using the gradient expansion,[6] which holds, since $\xi_B k_F \gg 1$. The expression for the current in the inhomogeneous order parameter field can be obtained in terms of the phase of the order parameter:[7]

$$\vec{j}(\vec{r}) = \frac{1}{2}\sum_{q,\vec{k}} \vec{k} n_q(\vec{k},\vec{r})\frac{\partial}{\partial \vec{r}}\Phi(\vec{k},\vec{r}) + \frac{1}{2}\frac{\partial}{\partial r_i}[\ \sum_{q,\vec{k}} \vec{k} n_q(\vec{k},\vec{r})\frac{\partial}{\partial k_i}\Phi(\vec{k},\vec{r})\ ]$$

$$-\frac{1}{2}\sum_{q,\vec{k}} \vec{k} n_q(\vec{k},\vec{r})\ (\frac{\partial}{\partial \vec{r}}\cdot\frac{\partial}{\partial \vec{k}} - \frac{\partial}{\partial \vec{k}}\cdot\frac{\partial}{\partial \vec{r}})\ \Phi(\vec{k},\vec{r}) \quad , \tag{14}$$

$$n_q(\vec{k}) = \frac{1}{2}(1 - \frac{\varepsilon_q(\vec{k}) - \mu}{E_q(\vec{k},\vec{r})}) \quad , \quad \tan\ \Phi(\vec{k},\vec{r}) = \frac{\vec{k}\cdot\hat{e}^2(x)}{\vec{k}\cdot\hat{e}^1(x)} . \tag{15}$$

The first term in Eq. (14) does not contribute to the current along y. The integration of the second term, which is a full derivative, over x from $-\infty$ to $+\infty$, leads to the regular contribution

$$\vec{J}_{\mathrm{regular}} = -\frac{\hbar}{4}\hat{y}\ \left(\rho(x_2) - \rho(x_1)\right), \tag{16}$$

This contribution, which exists even in the absence of the edge chiral states, is related to the edge currents produced by the orbital momentum of Cooper pairs $\vec{L} = \frac{1}{2}\hbar\rho\hat{l}$ (it is $\hbar\hat{l}$ for each pair of ^{3}He atoms of this p-wave superfluid):

$$\vec{J}_{\mathrm{regular}} = \frac{1}{2}\int_{-\infty}^{+\infty} dx\ \vec{\nabla}\times\vec{L}. \tag{17}$$

The third term, which is concentrated in the domain boundaries, gives the contribution from the chiral edge states. Since

$$n_q(\vec{k},\vec{r})\ \left(\frac{\partial}{\partial\vec{r}}\cdot\frac{\partial}{\partial\vec{k}} - \frac{\partial}{\partial\vec{k}}\cdot\frac{\partial}{\partial\vec{r}}\right)\Phi(\vec{k},\vec{r})$$
$$= \mathrm{sign}(k_y)\ 2\pi[\delta(x-x_1)-\delta(x-x_2)]\delta(k_x)\Theta(k_{qF}-k)\quad , \tag{18}$$

we obtain the following expression for this anomalous contribution to the current:

$$\vec{J}_{\mathrm{an}}(x_2) + \vec{J}_{\mathrm{an}}(x_1) = \hat{y}\ \frac{1}{2\pi\hbar}\ \sum_q(\mu(x_2)-\varepsilon_q(0))\Theta(\mu(x_2)-\varepsilon_q(0))$$
$$-\hat{y}\ \frac{1}{2\pi\hbar}\sum_q(\mu(x_1)-\varepsilon_q(0))\Theta(\mu(x_1)-\varepsilon_q(0)) = \hat{y}\frac{N}{2\pi\hbar}(\mu(x_2)-\mu(x_1))\quad , \tag{19}$$

where $N = 2_{q0}$ ($N = q0$ for the ^{3}He-A_1 film).

The anomalous current can be also obtained directly from the exact spectrum of the chiral mode in the vicinity of the Fermi point in Eq. (13). The change $\delta\mu$ in the chemical potential leads to a flow of the fermionic levels k_y through the Fermi points, and therefore the linear momentum $\propto\delta\mu$ is created from the vacuum. The response of the anomalous current $\hat{y}\Sigma_{q,k_y}k_y\Theta\ (-E_0(q,k_y))$ to $\delta\mu$ is thus

$$\frac{dJ_{\mathrm{an}}}{d\mu} = \frac{m_3}{2\pi\hbar}\sum_q\Theta(\mu-\varepsilon_q(0)) = N\frac{m_3}{2\pi\hbar}\quad , \tag{20}$$

which is the variation of Eq. (19). This momentum creation is the manifestation of the same chiral anomaly which was discussed for the bulk ^{3}He-A.[8,9]

The total current in the geometry of Fig. 1b is the sum of the regular and anomalous terms, Eq. (16) + Eq. (19), which leads to Eq. (2). In the geometry of Fig. 1a, there is no $\hat{l}$ outside the layer, and the regular contribution from Eq. (17) is absent. As for the anomalous contribution, one should retain only half of it, since the edge of the ^{3}He-A contains only half the number of the chiral fermions; this leads to Eq. (1). In the geometry of Fig. 1c, there are no edge states at all and one has only the regular contribution in Eq. (3); this case, which is considered in Ref. 4, has no exact quantization, although in the limit of the weak interaction between the fermions the quantity $\partial\rho/\partial\mu$ approaches stepwise behavior. The exact quantization of the current is related only to the chiral edge states. From Fig. 1 and Eqs. (1–3) it follows that the current obeys the summation rule, which follows from that for the Cooper pair orbital momentum: in particular, $\vec{J}^{(2)} + \vec{J}^{(3)} = 2\vec{J}^{(1)}$.

Note that the response of the current in this analog of QHE is quantized in terms of the same topological number N as the Chern-Simons term in the ^{3}He-A film, which determines the spin and quantum statistics of the particle-like solitons.[19]

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