# Higher-order Fourier approximations and exact algebraic solutions in the theory of the hydrodynamic Rayleigh–Taylor instability

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The asymptotic behavior of the hydrodynamic Rayleigh–Taylor instability is analyzed. A solution is constructed through a Fourier-series expansion. A system of equations for the amplitudes of the harmonics up to the sixth, inclusively, is constructed. This system of algebraic equations, with a high-order nonlinearity, can be solved exactly.

Cavitation, inertial fusion, some promising energy programs, the physics of high energy densities, and astrophysical phenomena are just a few of the applications in which Rayleigh–Taylor instabilities perform an important function.[1–11] Famous researchers such as Rayleigh, Ryabushinskiĭ, Taylor, and Birkhoff have analyzed this instability. The theory of the steady-state stage in the development of this instability is important to this analysis. Although this theory is the subject of active research,[12–19] the question of the steady state is still basically unclear. First, no converging sequence of approximations has yet to be offered; the question of whether such a sequence even exists remains open. Second, whether the solution is a point solution or a one-parameter solution is still under debate. Davies and Taylor,[13] Layzer,[14] and certain other authors believe that the solution is a point solution. According to Refs. 15, 16, 18, and 19, it is instead a one-parameter solution. The suggestion[15] that the solution is a one-parameter solution has remained a hypothesis, unsupported by a single quantitative argument.

In the present letter we propose some convincing quantitative arguments which lead to the conclusions that (first) solutions exist and (second) these solutions (I) form a one-parameter family and (II) are a unique such family.

The problem is to seek an analytic complex potential $f(z)$, $z = x + iy$, which satisfies the kinematic and dynamic boundary conditions.[14–17] We expand the potential $f$ in a Fourier series:

$$f(z;\vec{A}) = \sum_{n=1}^{N}(e^{inz} - 1 - inz)A_n/n, \quad f = \varphi + i\psi, \quad \vec{A} = A_1, \ldots, A_N. \tag{1}$$

We direct the $y$ axis opposite $\vec{g}$. We adopt $1/k$ as the unit of length, and $1/\sqrt{gk}$ as the unit of time. We "calibrate" $\psi$ and $p$ in such a way that the null streamlines $\psi(x,y;\vec{A}) = 0$ and also the null isobar $p(x,y;\vec{A}) = 0$ pass through the stagnation point $z = 0$. The expansions of the functions $\psi$ and $p$ which follow from (1) are

  

$$\psi = \sum_{s=0}^{\infty}\sum_{p=0}^{\infty} \psi_{sp} x^{2s} y^p; \quad p = \sum_{s=0}^{\infty}\sum_{p=0}^{\infty} p_{sp} x^{2s} y^p, \tag{2),(3}$$

$$\psi_{sp} = (-1)^{s+p} M_{2s+p}/(2s+1)!/p!, \quad M_k = \sum_{n=1}^{N} n^k A_n,$$

$$p_{sp} = \frac{(-1)^{s+p}}{(2s)!p!}\Big[\sum_{i=0}^{2s}\sum_{j=0}^{p}(-1)^i C_{2s}^i C_p^j M_{2s+p-i-j} M_{i+j} - 2M_{2s+p} M_0\Big],$$

$$\psi_{00} = 0, \quad p_{00} = 0, \quad p_{01} = 2, \quad C_N^n = N!/n!/(N-n)!$$

We transform the equations $\psi = 0$ and $p = 0$, which implicitly specify the null stream function and the null isobar, to the explicit form $y = y_\psi(x;\vec{A})$ and $y = y_p(x;\vec{A})$, respectively. The boundary conditions are satisfied if the following equations hold at a fixed $\vec{A}$:

$$y_\psi(x;\vec{A}) \equiv y_p(x;\vec{A}); \quad y_\psi = \sum_1 \psi_n x^{2n}, \quad y_p = \sum_1 p_n x^{2n}. \tag{4}$$

From (4) follows the system of algebraic equations which we are seeking:

$$\psi_n = p_n; \quad \vec{\psi} = \vec{\psi}(\hat{\psi}) = \vec{\psi}[\hat{\psi}(\vec{M})] = \vec{\psi}\{\hat{\psi}[\vec{M}(\vec{A})]\} = \vec{\psi}(\vec{A}), \quad \vec{p} = \vec{p}(\vec{A}), \tag{5}$$

where $\vec{\psi} = \psi_1, \ldots$, $\vec{p} = p_1, \ldots$, $\hat{\psi} = \{\psi_{sp}\}$, $\hat{p} = \{p_{sp}\}$; and $\vec{M} = M_0, \ldots$. The functions $\psi_n(\hat{\psi})$ and $p_n(\hat{p})$ are identical functions of their arguments $\psi_{sp}$ and $p_{sp}$, respectively. The functions $\psi_n$, $n = 1, 2, 3, 4, 5$, contain homogeneous forms of the arguments $\psi_{sp}$ of order $(2n-1)$. These functions consists of 1, 3, 9, 21, and 61 terms, respectively, after such forms are reduced. These expressions are very lengthy, so we will not reproduce them here.

Replacing $\hat{\psi}$ and $\hat{p}$ in $\psi_n$ and $p_n$, respectively, by their expressions in (2) and (3) in terms of the moments $\vec{M}$, and equating them in accordance with (5), we find the following equations, after some tedious manipulations:

$$\vec{U}(\vec{M}) = \vec{U}[\vec{M}(\vec{A})] = \vec{U}(\vec{A}) = 0, \quad \vec{U} = \{U_1, U_2, U_3, U_4, U_5\}, \tag{6}$$

$$U_1 = 3X^3 - Y, \quad U_2 = -90M_3XY + 9M_4X^2 + 95Y^3,$$

$$U_3 = -189M_3M_4X^3 + 27M_6X^4 - 567M_5X^3Y + 2394M_4X^2Y^2 - 2345Y^5,$$

$$\begin{aligned} U_4 = {} & 11340M_3^2M_4X^4 - 3402M_4M_5X^5 - 1620M_3M_6X^5 \\ & + 135M_8X^6 - 37800M_3^3X^3Y \\ & + 27216M_4^2X^4Y - 4860M_7X^5Y + 35910M_3M_4X^3Y^2 + 37530M_6X^4Y^2 \\ & + 157500M_3^2X^2Y^3 - 45990M_5X^3Y^3 - 261765M_4X^2Y^4 \\ & - 259350M_3XY^5 + 434000Y^7, \end{aligned}$$

$$U_5 = 38971625Y^9 - 243X^8M_{10} - 12843600XY^7M_3 - 4088700X^2Y^5M_3^2 + 6237000X^3Y^3M_3^3 - 1247400X^4YM_3^4 - 37525950X^2Y^6M_4 + 22609125X^3Y^4M_3M_4 - 9854460X^4Y^2M_3^2M_4 + 374220X^5M_3^3M_4 + 4571721X^4Y^3M_4^2 + 1347192X^5YM_3M_4^2 - 56133X^6M_4^3 - 4480245X^3Y^5M_5 + 415800X^4Y^3M_3M_5 + 1122660X^5YM_3^2M_5 + 149688X^5Y^2M_4M_5 - 224532X^6M_3M_4M_5 + 2133945X^4Y^4M_6 - 873180X^5Y^2M_3M_6 - 53460X^6M_3^2M_6 - 368874X^6YM_4M_6 + 16038X^7M_5M_6 + 516780X^5Y^3M_7 + 16038X^7M_4M_7 - 164835X^6Y^2M_8 + 4455X^7M_3M_8 + 13365X^7YM_9.$$

Perhaps the most striking result is that the huge and highly nonlinear system of equations in (6) turns out to be exactly solvable. We present here the solutions of these equations for the cases of $N=2$, $N=3$, $N=4$, and $N=5$ harmonics. It turns out that the nonlinearity is dominated by the first two moments: $X=M_1$ and $Y=M_2$.

We introduce the change of variables $X = -1/W/\sqrt{R}$, $Y = -3/W/R/\sqrt{R}$. The quantities $W$ and $R$ are the bubble ascent velocity and the radius of curvature of the boundary at the points $z=0$, respectively.

In terms of the moments $M_n$, $n>2$, the order of the equations, $U_k$ with index $k$, is equal to $k$. It thus becomes possible to construct an algebraic procedure for successively eliminating the higher moments.

The solutions for $N=2$ and $N=3$ are, respectively,

$$W = \frac{3(R-1)}{2R\sqrt{R}}, \quad W = \frac{135(4R^3 - 27R^2 + 36R - 19)}{(324(R-5)R)^{5/2}}. \tag{7),(8}$$

In the case $N=4$ the solution is found from the equation

$$a(R)W^2 + b(R)W + c(R) = 0, \tag{9}$$
$$a = 4032R^7, \; b = 120R^{7/2}(4389 - 5509R + 2385R^2 - 477R^3 + 20R^4),$$
$$c = -175(-3618 + 14325R - 24570R^2 + 23970R^3 - 13190R^4 + 3861R^5 - 558R^6 + 24R^7).$$

For $N=5$, the solution is found from a cubic equation by the Cartan formulas. This equation is

$$AW^3 + BW^2 + CW + D = 0, \tag{10}$$
$$A = -2764800R^{27/2}(21+R)^2(-21+5R),$$
$$B = 80640R^9(224056224 - 399930111R + 298384050R^2 - 114697920R^3 + 22361580R^4 - 2159139R^5 + 74666R^6 + 6150R^7),$$

TABLE I.

| $R$ | 2.01 | 2.21 | 2.61 | 3.41 |
|---|---|---|---|---|
| $W(4)$ | 0.5604 | 0.5765 | 0.5997 | 0.6252 |
| $W(5)$ | 0.5638 | 0.5776 | 0.6004 | 0.6274 |
| $a_1(4)$ | -0.7747 | -0.8552 | -0.9820 | -1.1447 |
| $a_1(5)$ | -0.8013 | -0.8748 | -1.0185 | -1.2470 |
| $a_2(4)$ | -0.2099 | -0.1427 | -0.0347 | 0.0847 |
| $a_2(5)$ | -0.1484 | -0.0889 | 0.0723 | 0.3863 |
| $a_3(4)$ | 0.0025 | 0.0177 | 0.0475 | 0.1310 |
| $a_3(5)$ | -0.0508 | -0.0343 | -0.0591 | - 0.1723 |
| $a_4(4)$ | -0.0179 | -0.0198 | -0.0308 | - 0.0709 |
| $a_4(5)$ | 0.0031 | -0.00092 | 0.0062 | 0.0369 |
| $a_5(5)$ | -0.0026 | -0.0011 | -0.0009 | - 0.0039 |

$$C = 2352R^{9/2}(-242330493741 + 1077546374055R - 2102913782685R^2 +2378472067535R^3 - 1737136829925R^4 + 858834704067R^5 -291009641115R^6 + 66242771685R^7 - 9503032410R^8 + 750812250R^9 -19358676R^{10} - 1113840R^{11} + 28000R^{12}),$$

$$D = -343(-1233120806973 + 10258653190050R - 39590410528134R^2 +93960585281430R^3 - 153346137297426R^4 + 181575828795132R^5 -159859650164000R^6 + 105490798752108R^7 - 52083469287405R^8 +19054230016122R^9 - 5066474615490R^{10} + 944934061950R^{11} -115544972124R^{12} + 8014545000R^{13} - 177358896R^{14} -10278144R^{15} + 345600R^{16}).$$

The results calculated from (7)–(10), after a selection of roots and a calculation of the amplitudes $\vec{A}$, are listed in Table I. The index in parentheses is the value of $N$. The amplitudes $\vec{A}$ are normalized to the velocity: $\vec{a} = \vec{A}/W$. Analysis shows that there is a fast, roughly exponential convergence. This result indicates that a solution exists. An examination of all the existing roots—such an examination is possible in this case because we have derived a complete algebraic solution—shows that the converging solution is unique. The converging solutions form a one-parameter family. The parameter $\lambda$, which runs over the memebers of this family, lies between certain limits. If we choose the radius $R$ as $\lambda$, we find that the approximations converge for $R_{crit} < R < R_{max}$, where $R_{crit} = 1.97 \pm 0.07$, and $R_{max}$ is determined by the number of harmonics ($N$) incorporated. Specifically, it increases with increasing $N$.

The calculations were carried out by a program for analytic conversions by computer,[20] on a personal computer of the PC AT 386/7 type, with 8 Mbytes of memory.

The appearance of polynomial equations is typical of Galerkin approximations of dynamic systems. The degree of the polynomials increases rapidly with improving quality of the approximation. The results found here, which are closely related to the Grebner method for solving such equations, indicate that the approximating equations can be solved exactly even for fairly high orders of the approximation.

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