Quantum birth of a universe near a cosmological singularity

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The example of the homogeneous Bianchi IX model filled with a scalar field is used to show that universes are born in the course of a quantum evolution from the vacuum state. In the limit in which this scalar field has a small energy density, $\epsilon \to 0$, the spectral number density of universes has a Planckian shape.

In classical cosmology the time over which a universe exists is limited by a singularity (a singular point in time). At this singularity, classical theory breaks down and must be replaced by a quantum theory. One of the most interesting problems in quantum cosmology is that of describing the quantum production of a world from nothing.[1,2] A process of this sort can be described correctly by a theory which permits a variable number of universes. In other words, the wave function of the universe must be second-quantized (from the standpoint of particles, this approach corresponds to a third quantization). The idea of using third quantization to describe the birth of a universe is not new; it was discussed in Ref. 3 in the example of an isotropic model. The possibility that universes are born stems from the circumstance that the Wheeler–de Witt equation, which the wave function must satisfy, is a second-order equation of the Klein–Gordon type and contains an explicit time dependence (if one of the metric functions is chosen as the time). Below we examine a quantum birth of a universe in the example of a homogeneous model of type IX, filled with a massless scalar field φ. The reason for choosing this model is that in classical theory it describes the local behavior of the overall nonuniform gravitational field near a singularity.[4] The only type of matter which need be taken into account in the leading order is the scalar field.[5] One might expect that a situation of this sort would also hold in quantum cosmology.

The metric for the homogeneous Bianchi IX model is

$$ds^2 = -N^2 dt^2 + \sum e^{q_l} \sigma^l \sigma^l, \tag{1}$$

where $\sigma^l = \sigma^l_j(x) dx^j$ are uniform basis forms, and the q_l are conveniently written in the form[6] $q_l = \Omega + 2\,\mathrm{Re}(q \exp(i\theta_l))$, where $\theta_l = 0, \pm 2\pi/3$, Ω is proportional to the logarithm of the volume of the space, and the complex variable q characterizes the degree of anisotropy of the space.

In quantum cosmology the state of the universe is described by the wave function Ψ, which is a functional specified on a superspace W, which is the space of all metrics and all matter fields. The evolution of Ψ is governed by the Wheeler–de Witt equation[7]

$$(\Delta + U)\Psi = 0, \tag{2}$$

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where $U = 6\lambda\,(\Sigma\, e^{2q_l} - \frac{1}{2}\Sigma\, e^{q_l})^2)$, $\Delta = (1/\sqrt{-G})\partial_A\sqrt{-G}G^{AB}\partial_B$, G_{AB} is the metric on W, defined by the interval $d\Gamma^2 = \frac{1}{\lambda}\,(d\varphi^2 + |dq|^2 - d\Psi^2)$, and $\lambda = \frac{2}{3}Ne^{-3\Omega/2}$. Equation (2) can formally be derived from the action

$$S = \frac{1}{2}\int (G^{AB}\partial_A\Psi^*\partial_B\Psi - U|\Psi|^2)\sqrt{-G}d^4\zeta,$$

where ζ^A are the coordinates on W. In terms of the coordinates $\Omega = -e^{-\tau}(1+|z|^2/1-|z|^2)$, $q = e^{-\tau}(2z/1-|z|^2)$, the metric on W takes the form

$$d\Gamma^2 = \frac{1}{\lambda}\left[(d\varphi)^2 + e^{-2\tau}\left(\frac{4dzd\bar{z}}{(1-|z|^2)^2} - (d\tau)^2\right)\right]. \tag{3}$$

Making use of the latitude in the choice of the course function N, we set $\lambda = 1$. We consider the asymptotic situation as $\tau \to -\infty$ $(\Omega \to -\infty)$, which corresponds to a singularity in metric (1). In this case the potential in (2) can be modeled by infinite walls:[7]

$$e^{q_l} = \exp(-e^{-\tau}\eta_l(z)) \xrightarrow[\tau\to-\infty]{} \begin{cases} 0 & \text{for } \eta_l(z) > 0 \\ \infty & \phantom{\text{for }} \eta_l(z) < 0 \end{cases},$$

$$\eta_l = \frac{1+|z| - 4\mathrm{Re}\,z\exp(i\theta_l)}{1-|z|^2}.$$

In this limit, U depends on z alone. The functions $u_p(\Omega,z) \sim \exp(\tau/2)\chi_p(\tau)\varphi_n(z)\exp(i\epsilon\varphi)$ then constitute a complete orthonormal set of solutions of Eq. (2). Here $p = (n,\epsilon)$; φ_n satisfies the equation $\Delta_z\varphi_n = -(k_n^2 + 1/4)\varphi_n$ with the boundary conditions $\varphi_n(z) = 0$ at $\eta_l(z) = 0$ and with a Laplacian Δ_z constructed with the help of the metric $4dzd\bar{z}/(1-|z|^2)^2$; and χ_p satifies the equation

$$\frac{d^2\chi_p}{d\tau^2} + (k_n^2 + \epsilon^2 e^{-2\tau})\chi_p = 0 \tag{4}$$

and is expressed in terms of Bessel or Hankel functions.

When a third quantization is imposed, the wave function of the universe becomes a field operator and can be expanded in an arbitrary complete orthonormal set $\{u_p\}$ of solutions of Eq. (2) (for simplicity we assume below that Ψ is real): $\Psi = \Sigma\, a_p u_p + a_p^+ u_p^*$. The operators a_p and a_p^+ satisfy the commutation relations $[a_p, a_{p'}^+] = \delta_{p,p'}$. As we mentioned earlier, Eq. (2) has an explicit time dependence, so there is no unambiguous classification of the solutions $\{u_p\}$ in terms of the sign of the frequency. A corpuscular interpretation can be generated either by diagonalizing the Hamiltonian[9] or by singling out the asymptotic "in" and "out" regions on W, for which we can determine positive-frequency solutions of (2). If such regions exist, these two methods agree.[9] We determine various asymptotic regions: in $(\epsilon e^{-\tau} \gg k_n)$, which corresponds to a physical singularity, and out $(\epsilon e^{-\tau} \ll k_n)$. The latter can be reached only as $\epsilon \to 0$, since we are considering the vicinity of the singularity. The positive-frequency modes in the "in" and "out" regions are

$$\chi_p^{\text{in}}(\tau) = \frac{1}{2}(\pi)^{1/2}\exp(\pi k_n/2) H_{ik_n}(\epsilon t),$$

$$\chi_p^{\text{out}}(\tau) = [(2/\pi)\sinh(\pi k_n)]^{1/2} J_{ik_n}(\epsilon t), \quad t = e^{-\tau}. \tag{5}$$

These modes are related by a Bogolyubov transformation $\chi_p^{\text{out}} = \alpha_p \chi_p^{\text{in}} + \beta_p \chi_p^{*\text{in}}$ with the coefficients

$$\alpha_p = [\exp(\pi k_n)/2\sinh(\pi k_n)]^{1/2}, \quad \beta_p = [\exp(-\pi k_n)/2\sinh(\pi k_n)]^{1/2}.$$

If we choose the vacuum associated with the "in" modes in (5), $|0, \text{in}\rangle$, as the initial state, we can calculate the expectation value of the number of universes present in the "out" modes:

$$N_p = < 0, \text{in}|a_{p,\text{out}}^+, a_{p,\text{out}}|0, \text{in} > = |\beta_p|^2 = (\exp(2\pi k_n) - 1)^{-1}. \tag{6}$$

This value is independent of ϵ and has a Planckian form with a temperature $T_0 = 1/2\pi$. The temperature T perceived by an "observer" is given by the Tolman relation $T = (G_{00})^{-1/2} T_0$, where G_{00} is a component of the metric on W in which the "observer" has a constant coordinate [for metric (3) we would have $T = (1/2\pi)\exp\tau$].

The quantities k_n in (6) represent the energy of the gravitational field associated with the anisotropy of the space. The anisotropy parameters z can be interpreted as amplitudes of gravitational waves of various polarizations with the maximum possible wavelength.[6] Expression (6) thus gives the distribution of the number of universes with respect to the energy of the purely gravitational field.

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