Quasiadiabatic model of charged-particle motion in a dipole magnetic confinement system under conditions of dynamic chaos

V. D. Il'in, S. N. Kuznetsov, and B. Yu. Yushkov
Scientific-Research Institute of Nuclear Physics, Moscow State University, 119899, Moscow

I. V. Il'in
Troitsk Institute of Innovative and Thermonuclear Research, 142092, Troitsk, Moscow Oblast

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A model is constructed for the motion of nonadiabatic particles in a region of stochastic instability. This model differs from the standard adiabatic theory in that the particle spirals around a trajectory which passes through the center of the dipole (the "central trajectory"), rather than around a field line. There exists a constant of the motion, which is an analog of the magnetic moment in the adiabatic model. The equations for transforming from the coordinate system of the field line to that of the central trajectory are given. The transformations between coordinate systems are Euler rotations in the direction in which the particle revolves and drifts. These ideas are used to derive a simple Poincaré mapping for the coordinate system of the central trajectory. This mapping describes the long-term evolution of the particles.

The dynamics of a Hamiltonian system consisting of a particle and a dipole field is of research interest for several reasons, primarily because this is the simplest nonlinear system which exhibits a deterministic chaos. This circumstance is in turn related to the problem of the stability of particle motion in magnetic confinement systems, particularly (because of the dipole nature of the field) the magnetic confinement systems of many astrophysical objects.

The existing model of the motion is based mostly on the concept of the first adiabatic invariant (i.e., the magnetic moment of the particle, μ) and the degree to which this invariant is conserved as a function of the adiabatic parameter χ. Here we have $\mu \sim \sin^2\alpha/B$, where α is the angle between the velocity and the field, and B is the magnetic field. The adiabatic parameter is given by $\chi = \rho/R_c$, where $\rho = v_0/\omega$ is the total Larmor radius in the median (equatorial) plane, R_c is the radius of curvature of a field line, v_0 is the magnitude of the velocity vector, and ω is the Larmor frequency. A violation of the invariance of μ stems from the existence of resonances between oscillations of the particle in different degrees of freedom. This situation could ultimately result in the onset of a stochastic motion. This resonance approach was worked out in Ref. 1. Important here is the idea that μ changes "abruptly" as the equatorial plane is crossed. At sufficiently small values $\chi \ll 1$, we can indeed assume that we have $\mu_1 \simeq \mathrm{const}_1$ up to the equator and then $\mu_2 \simeq \mathrm{const}_2$ beyond it. It is then a simple matter to find the positions of the bounce points. Even at $\chi \gtrsim 0.1$, however, the picture becomes much more complicated. The day is not saved by refined expressions for μ which reflect subsequent terms in the asymptotic series for the magnetic moment.[6]

Everything does simplify, and radically, if the motion of an arbitrary particle is examined not with respect to a magnetic field line but with respect to the corresponding central trajectory. Let us outline a computational algorithm for finding the central trajectory and some of its properties. The "central trajectory" is the trajectory traced out by a particle which is injected from the center of the dipole along a given field line, i.e., at an angle $\alpha = 0$. Computer limitations force us to select the injection point on a given field line, at a distance from the center of the dipole such that its position has essentially no effect on the results of the calculation. As it moves along the central trajectory, the particle undergoes essentially no revolution, since the velocity of the Larmor revolution is $v_\perp = v_0 \sin\alpha \simeq v_{dr}$, where v_{dr} is the magnetic (transverse) drift of the particle. The values of the phase ($\tilde{\varphi}_0$) and the angle ($\tilde{\alpha}_0$) with which the particle arrives at the equator depend on only the parameter χ and can be approximated by the functions

$$\tilde{\alpha}_0 \simeq 2.1\chi_i^{0.26} - 1.19, \quad \tilde{\varphi}_0 \simeq 1.95\chi_i^{0.3} + 0.38, \quad 0.24 \le \chi_i \le 1.36, \tag{1}$$

where $\chi_i = \chi(R_{ei})$, $R_{ei} = R_i \cos^{-2}\lambda_i$, R_i is the distance from the dipole to the injection point, R_{ei} is the distance from the dipole to the point at which the given field line intersects the equatorial plane, and λ_i is the angular distance from the equator to the injection point (the latitude of the injection point). The angles $\tilde{\alpha}_0$ and $\tilde{\varphi}_0$ are expressed in radians. The magnitude of the equatorial radius vector of the particle, R_e, is related to $\tilde{\alpha}_0$ and $\tilde{\varphi}_0$ through an exact integral of motion (a generalized momentum), which gives us, in this case,

$$R_e = \frac{2\gamma R_{ei}}{\gamma + \sqrt{\gamma^2 - \sin\tilde{\alpha}_0 \sin\tilde{\varphi}_0}}, \quad 2\gamma = \left(\frac{3}{\chi_i}\right)^{1/2}. \tag{2}$$

The excursion of the central trajectory from the field line is greatest at the equator. It naturally decreases with decreasing χ_i in accordance with (2) ($R_e \to R_{ei}$ as $\chi_i \to 0$).

Further on (beyond the equator) the particle spirals as it moves along the central trajectory, and it is reflected, depending on χ_i, at $R > 0$. However, if the phase of this particle, which has arrived at the equator, is turned to $\pi - \tilde{\varphi}_0$, while the angle $\tilde{\alpha}_0$ is not changed, the particle will move along a central trajectory on the other side of the equator. The coordinate system obtained in this manner consists of a pre-equator central trajectory and a post-equator central trajectory, the two being characterized by the values $\tilde{\varphi}_{01}$, $\tilde{\alpha}_0$ and $\tilde{\varphi}_{02}$, $\tilde{\alpha}_0$ at the equator. Regardless of the value of χ, we have $\tilde{\varphi}_{01} + \tilde{\varphi}_{02} = \pi$; at $\chi < 0.13$ we have $\tilde{\varphi}_{01} \simeq \tilde{\varphi}_{02} \simeq \pi/2$. The angular distance between the points $\tilde{\varphi}_{01}$, $\tilde{\alpha}_0$ and $\tilde{\varphi}_{02}$, $\tilde{\alpha}_0$ is $\nu = 2\arcsin(\sin\tilde{\alpha}_0 \sin\delta)$, where $\delta = \pi/2 - \tilde{\varphi}_{01} = \tilde{\varphi}_{02} - \pi/2$.

We now introduce $\mu^* \sim \sin^2\alpha^*/B$ along with μ, where α^* is the angle between the velocity vector and the tangent to the central trajectory. Figure 1 shows the behavior of μ and μ^* along the trajectory of the particle (a trajectory other than the central trajectory), as found through a numerical integration of the equations of motion between two bounce points. We see that μ changes markedly, while μ^* remains essentially constant to the left and right of the equator. Only at $\lambda = 0$ do we find the abrupt

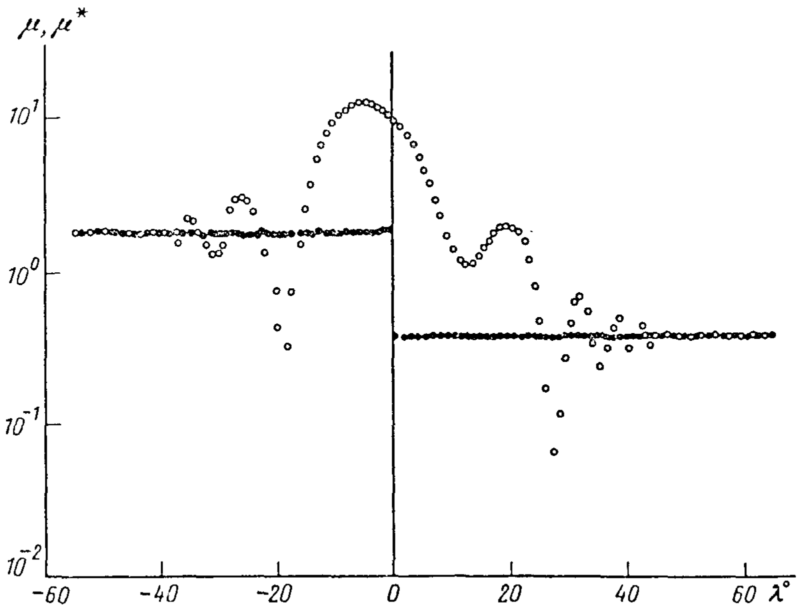

FIG. 1. Plots of $\mu \sim \sin^2 \alpha / B$ (open circles) and $\mu^* \sim \sin^2 \alpha^* / B$ (filled circles) versus the latitude λ during the motion of a particle with an energy $W = 200$ MeV between bounce points in the dipole geomagnetic confinement system. At the equator ($\lambda = 0$) the following values prevail: $R_{ei} = 2.9 R_{\odot}$ ($R_{\odot}$ is the earth's radius); $\chi_i = 0.272$; $\alpha = 20°$; $\phi = 108°$.

change in μ^* (and, correspondingly, in α^*), as discussed above. The magnitude of this abrupt change, $\Delta\mu^* = \mu_{m_1} - \mu_{m_2} = \Delta\mu = (\Delta\mu/\mu)\mu$, agrees within $\sim 10\%$ with the theoretical estimate of $\Delta\mu$ found in Ref. 2. Here μ_m is the value of μ at the bounce point. The size of abrupt change in α^* is

$$\Delta\alpha^* \simeq \nu \cos\varphi \simeq \frac{1}{2} \tan\alpha \cdot \left(\frac{\Delta\mu}{\mu}\right). \tag{3}$$

In the new coordinate system, the z' axis coincides with the tangent to the corresponding central trajectory, and at the equator it is determined by the parameters $\tilde{\alpha}_0$ and $\tilde{\varphi}_0$. A transformation from the usual coordinate system, in which the z axis runs along the tangent to B (and we have $z \perp R_e$ at the equator), to the new coordinate system is made by rotating the initial system an angle $\tilde{\varphi}_0$ in the drift direction around the z axis and by then bringing the z and z' axes into coincidence (by means of a rotation through an angle $\tilde{\alpha}_0$). The relations between coordinate systems can be written

$$\cos\alpha = \cos\alpha^* \cos\tilde{\alpha}_0 - \sin\alpha^* \sin\alpha_0 \cos\phi, \tag{4}$$
$$\sin(\varphi - \tilde{\varphi}_0) = \sin\alpha^* \sin\phi / \sin\alpha, \quad 0 \le \phi \le 2\pi.$$

Upon each crossing of the equator, the angular coordinates of the particle in the new coordinate system change in accordance with the recurrence relations

$$\alpha^*_n, \phi_n \to \alpha^*_{n+1}, \phi_{n+1}, \tag{5}$$

where $\alpha^*_{n+1} = \alpha^*_n + \Delta\alpha^*_n \cos\phi_n$ and $\phi_{n+1} = \phi_n + \Delta\phi(\alpha^*_{n+1})$. The change in phase (the phase shift) between two successive crossings of the equatorial plane is given approximately by[3]

$$\Delta\phi = \frac{\pi\bar{\omega}}{\Omega} = \frac{6F(\alpha)}{\chi}, \quad F(\alpha) \approx \sin^{-1,348}\alpha - 0.255, \quad 1.25^0 \leq \alpha \leq 90^0, \tag{6}$$

where $\overline{\omega}$ is the Larmor frequency averaged over a longitudinal oscillation, and Ω is the frequency of the longitudinal oscillations. Using (3)–(6), we finally find the mapping

$$\cos\alpha^*_{n+1} = \cos\alpha^*_n \cos\nu + \sin\alpha^*_n \sin\nu \cos(\phi_n - \tilde{\phi}_0), \tag{7}$$

$$\phi_{n+1} = \phi_n + \frac{6F(\alpha_{n+1})}{\chi_i},$$

where $\tilde{\phi}_0 = \arcsin(\sin\tilde{\alpha}_0 \sin 2\delta / \sin\nu)$. By analogy with Ref. 1, we can put recurrence relations (5) in the form of a standard Chirikov mapping:

$$I^*_{n+1} = I^*_n + K^* \sin\theta_n, \quad \theta_{n+1} = \theta_n + I^*_{n+1}, \tag{8}$$

where $I^* = 6/\chi(\partial F/\partial\alpha^*)_{\alpha^*_r} \cdot (\alpha^* - \alpha^*_r)$, $K^* = 6\nu/\chi(\partial F/\partial\alpha^*)_{\alpha^*_r}$, $\theta = \phi - \tilde{\phi}_0 - \pi/2$, and α^*_r is found from the resonance condition $\overline{\omega} = 2r\Omega$ (r is an integer). As a result, we find that this model of the motion incorporates an "adiabatic" motion ($\mu^* = \text{const}$) between the equator and the bounce point and also discrete model (7) [or (8)] for multiple longitudinal oscillations of the particle. This model works effectively under the conditions $\chi < 1$ and $\alpha^* \leqslant 45°$, which correspond to a region of stochastic instability.[3] As $\chi \to 0$, the central-trajectory model becomes the adiabatic model with the corresponding field line ($\alpha \to \alpha^*$, $\Delta\alpha^* \to 0$, $\alpha^*_{n+1} \to \alpha^*_n$, $\tilde{\alpha}_0$, and $\delta \to 0$).

To illustrate the effectiveness of this model, we calculated the capture coefficient $c_\odot$ and the proton confinement time τ_n in the dipole geomagnetic field for the case in which protons were injected from the so-called absorption cone (or loss cone[4]). This problem was solved in Ref. 5 through a numerical integration of the equations of motion. We restrict the discussion here to the case with the strongest statistical base, in which 744 trajectories were analyzed for protons with an energy $W = 4.3$ GeV. The protons were injected from the earth's surface with $R_{ei} = 1.5R_\odot$ ($\chi_i \simeq 0.584$), where $R_\odot$ is the earth's radius. The initial conditions α_0 and φ_0 were generated by the Monte Carlo method. In terms of our variables, the equatorial loss cone (which is also the region from which the entire set of particles in injected) is defined by the interval

$$0 \leqslant \sin\alpha^* \leqslant \sin\alpha^*_c = [(4R_{ei}/R_\odot - 3)(R_{ei}/R_\odot)^5]^{-1/4},$$

as in the adiabatic theory. A particle was assumed to exit the loss cone if it had an angle $\alpha^*_1 > \alpha^*_c$ after the first iteration [see (7)]. The particle could subsequently be absorbed, if a value $\alpha^*_n \leqslant \alpha^*_c$ arose. In both cases, we calculated $c_\odot = N_1/N_0$, where N_0

is the number of particles injected, and N_1 is the number of particles after the first iteration (in Ref. 5, it represented the particles which had at least one bounce point above the earth's surface). We also calculated τ_n, which corresponds to $N_1/N = \varepsilon$. We found $c_\odot = 0.77(0.74)$ and $\tau_n = 5(5)$, where τ_n is expressed in units of the number of bounces, and the numbers in parentheses are the results of our calculations from (7).

The model of the motion constructed here could in principle be used for magnetic confinement systems of a variety of configurations.

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