Dense (2+1)-dimensional QED in an external magnetic field

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(Submitted 30 April 1992)
Pis'ma Zh. Eksp. Teor. Fiz. **55**, No. 12, 673–675 (25 June 1992)

The effective action is derived for a QED_{2+1} with a nonzero chemical potential in an external magnetic field. The condition of electrical neutrality, $\partial L/\partial\mu = 0$, has a set of solutions. When this condition holds, the minimum of the energy density E is reached at $B \neq 0$.

Research on quantum field theory in a $(2+1)$-dimensional space has recently been attracting considerable interest.[1] The reason for this interest is that certain real effects in solids, e.g., the fractional Hall effect[2] and high T_c superconductivity,[3] can be described on the basis of a 2D theory. Since a nonzero charge density and the presence of a magnetic field are important here, we think it relevant to study a $(2+1)$-dimensional electrodynamics (QED_{2+1}) with a nonzero chemical potential in an external magnetic field.

The dense QED_{2+1} is described by the Lagrangian[1)]

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - eA\!\!\!/ - \gamma_0\mu - m)\psi\,, \tag{1}$$

where μ is the chemical potential, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $A\!\!\!/ = A_\mu\gamma^\mu$, $\gamma_0 = \sigma_3$, $\gamma_{1,2} = i\sigma_{1,2}$, and σ_i are the Pauli matrices.

Let us calculate the single-loop effective action $S^{\rm eff}$ of the dense QED_{2+1},

$$\begin{aligned} S^{eff} &= -i\ln\,\mathrm{Det}(i\partial\!\!\!/ - eA\!\!\!/ - \gamma_0\mu - m)\\ &= -\tfrac{i}{2}(\ln\,\mathrm{Det}(i\partial\!\!\!/ - eA\!\!\!/ - \gamma_0\mu - m) + \ln\,\mathrm{Det}(i\partial\!\!\!/ - eA\!\!\!/ - \gamma_0\mu + m))\\ &\quad - \tfrac{i}{2}(\ln\,\mathrm{Det}(i\partial\!\!\!/ - eA\!\!\!/ - \gamma_0\mu - m) - \ln\,\mathrm{Det}(i\partial\!\!\!/ - eA\!\!\!/ - \gamma_0\mu + m))\\ &= S^{eff}_{even} + S^{eff}_{odd}\,, \end{aligned} \tag{2}$$

in an external magnetic field B. This field is given by the vector potential $A_0 = A_2 = 0$, $A_1 = x_2 B$ [we should stress that this second term on the right side of (2) breaks P and T parity in QED_{2+1}; Refs. 1, 4, and 5].

In calculating $S^{\rm eff}_{\rm even} = \int d^3x L^{\rm eff}_{\rm even}$ we can make direct use of the Fock–Schwinger proper-time method, modified for the case of a dense medium:[6]

$$L^{eff}_{even} = \frac{1}{8\pi^{\frac{3}{2}}}\int_0^\infty \frac{ds}{s^{5/2}}\,e^{-m^2 s}(eBs\coth(eBs) - 1) \tag{3}$$

($L^{\rm eff}_{\rm even}$ is independent of μ and is the same as the result found in Ref. 4 in the case $\mu = 0$).

To calculate $S^{\rm eff}_{\rm odd} = \int d^3x L^{\rm eff}_{\rm odd}$, we first find the current $\int d^3x I_\mu = \delta S/\delta A^\mu$, and then reconstruct the corresponding term in the action.[4] The current can be expressed in terms of the fermion Green's function $G(x,y)$:

$$\int d^3x I_\mu = \frac{\delta S}{\delta A^\mu} = ie\mathrm{Tr}(\gamma_\mu G(x=y)) = \frac{ie}{(2\pi)^3}\int d^3p Tr\gamma_\mu G(p) . \tag{4}$$

For a dense QED_{2+1} in an external magnetic field B, this function is[2)]

$$\begin{aligned} &G(p,\mu) \\ &= -i\theta((p_0-\mu)\mathrm{sign}p_0)\int_0^\infty ds\{\not p' + m - (\gamma^1 p_2 - \gamma^2 p_1)\tan(eBs)\} \\ &\times(1 - i\sigma_3\tan(eBs))\exp\{is(p'^2_0 - \vec p^2\tfrac{\tan(eBs)}{eBs} - m^2 + i\varepsilon)\} \\ &+ i\theta(-(p_0-\mu)\mathrm{sign}p_0)\int_0^\infty ds\{\not p' + m + (\gamma^1 p_2 - \gamma^2 p_1)\tan(eBs)\} \\ &\times(1 + i\sigma_3\tan(eBs))\exp\{-is(p'^2_0 - \vec p^2\tfrac{\tan(eBs)}{eBs} - m^2 - i\varepsilon)\}, \end{aligned} \tag{5}$$

where $p' = (p_0 - \mu, \vec p)$, and 1 is the 2×2 unit diagonal matrix.

It is easy to see that the spatial components of the current I_j vanish after an integration over $\vec p$. In place of I_0 we consider the fermion density $\rho(B,\mu) = \partial L/\partial\mu$, which differs from I_0 by a factor of e. Using (5), we find

$$\begin{aligned} \rho(B,\mu) = \rho(B) + \tfrac{1}{4\pi^3}\int_0^\mu dx \int \vec p \int_0^\infty ds\{&(x + im\tan(eBs)) \\ &\times\exp\left\{is\left(x^2 - \vec p^2\frac{\tan(eBs)}{eBs} - m^2 + i\varepsilon\right)\right\} \\ &+ (x - im\tan(eBs))\exp\left\{-is\left(x^2 - \vec p^2\frac{\tan(eBs)}{eBs} - m^2 - i\varepsilon\right)\right\}\} . \end{aligned} \tag{6}$$

Here $\rho(B) = eB/4\pi$ corresponds to an anomalous current which arises in QED_{2+1} in an external field[4,7] and which is associated with an asymmetry of the fermion spectrum [the fermion Green's function in QED_{2+1} in the $\mu=0$ case has poles at $p_0 = -m$, $p_0 = \pm(m^2+2eBn)^{1/2}$, $n = 1,2,\ldots$ (Ref. 8); here and below, we are assuming eV, $m>0$].

Evaluating the integral in (6), we find the final result:

$$\rho(B,\mu) = \frac{eB}{4\pi} + \frac{\mu^2 - m^2}{4\pi}\theta(|\mu| - m)sign\mu + n\frac{eB}{2\pi} , \tag{7}$$

where

$$n = \begin{cases} -1 - [\frac{\mu^2 - m^2}{2eB}], & \mu > m; \\ 0, & |\mu| < m; \\ [\frac{\mu^2 - m^2}{2eB}], & \mu < -m \end{cases} \tag{8}$$

([...] means the greatest integer).

The last term in (7) arises when we circumvent the poles in the integrand in (6); these poles coincide with the poles of the Green's function in the $\mu = 0$ case[8] [expression (7) could also be derived by the method developed by Niemi[9]].

An important consequence of (7) and (8) is that in QED_{2+1} we could have a situation in which the medium is electrically neutral ($\partial L/\partial\mu = \rho = 0$) and the chemical potential is nonzero. Specifically, it would be $\mu = \pm [m^2 + (2n+1)eB]^{1/2}$, $n = 1,2,...$.

Reconstructing $L^{\rm eff}_{\rm odd}$, we find that, for $m < \mu < (m^2 + 2eB)^{1/2}$, for example, we have

$$L^{eff}_{odd} = \frac{1}{4\pi}\left(eBm + \frac{\mu^3 - m^3}{3} - m^2(\mu - m) - eB(\mu - m)\right). \qquad (9)$$

We now impose the condition of electrical neutrality ($\mu^2 - m^2 = eB$), and we take the limit $eB/m^2 \ll 1$ [as we must in order to evaluate the integral in (3)]. We find the following expression for the effective Lagrangian:

$$L^{eff} = \frac{e^2}{m}\frac{B^2}{24\pi} + \frac{eBm}{4\pi}. \qquad (10)$$

The energy density is correspondingly

$$\mathcal{E}(B) = -L = -L^{classical} - L^{eff} = \frac{B^2}{2}\left(1 - \frac{e^2}{12\pi m}\right) - \frac{|eB|m}{4\pi}. \qquad (11)$$

It follows from (11) that the energy reaches a minimum at $B \simeq \pm em/4\pi$ and $E_{\rm min} = -e^2m^2/32\pi^2 < 0$. This result means that a magnetization arises.

We have discussed only one solution of the electrical-neutrality equation; a corresponding analysis could be carried out for each such solution. The value of $E_{\rm min}$ is the same everywhere, because the term $eBm/4\pi$ is predominant in $L^{\rm eff}_{\rm odd}$ in our approximation.

In summary, we have shown that in a dense QED_{2+1} in an external magnetic field there can be a case in which the medium is electrically neutral despite the condition $\mu \neq 0$, and the energy reaches a minimum at $B \neq 0$.

I am deeply indebted to I. A. Batalin, D. A. Kirzhnits, O. I. Loĭko, V. V. Losyakov, V. V. Skalozub, and A. E. Shabad for discussions and valuable comments.

[1)]A Chern-Simons term $(\theta/4)\epsilon_{\mu\nu\alpha}F^{\mu\nu}A^{\alpha}$ could be added here.[1]

[2)]When the chemical potential is nonzero, the rule for circumventing the poles of the Green's function is altered. The corresponding generalization is not a trivial problem.[6]

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Translated by D. Parsons