Perturbation theory based on the physical branch of the renormalization-group equation

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A physical branch of the exact (local) renormalization-group equation which was recently found is used as a zeroth approximation for a verifiable gradient expansion.

Formally, the exact variational-derivative renormalization-group equation, which was proposed in the early studies of the fluctuation theory of phase transitions,[1,2] makes it possible in principle to find a fixed point exactly and to calculate critical exponents. In practice, however, a perturbation theory of some sort must be used. These perturbation theories either are based on sizeable parameters[3,4] or assume some

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uncontrollable criteria for cutting off the systems of coupled equations for the vertices of the free-energy functional.[5] The primary difficulty is that a rigorous calculation procedure should use a definitely nonlocal form of this functional.[6,7] The expected generation of nonlocal corrections is slight to the extent that the Fisher index is small ($\eta \approx 0.03$). One can thus work with a local version of the exact renormalization-group equation as a first step[8-12] and then carry out an expansion in nonlocal corrections.

The local renormalization-group equation, however, is strongly nonlinear, and a perturbation theory (various versions of the same ϵ expansion) had to be used to solve it. The use of a φ^4 local form as a zeroth approximation,[12] for example, requires an expansion of the difference $(4-d)$ in powers of $\eta^{1/2}$. Carrying out such an expansion is equivalent to modifying an ϵ expansion.[13] Nevertheless, the idea of a perturbation theory based on the small extent to which nonlocal corrections are generated remains attractive.

The (unique) physical branch of the solution of the local renormalization-group equation which was recently found[14,15] makes it possible to formulate the perturbation theory of interest in the form of a gradient expansion.[16] The exact variational-derivative renormalization-group equation is[17]

$$H[\varphi] = \frac{1}{2}\int_{\mathrm{q}} \eta(\mathrm{q})[V - G_0^{-1}(\mathrm{q})|\vec{\varphi}(\mathrm{q})|^2] + dV\frac{\partial H}{\partial V}$$
$$- \int d^{\mathrm{d}}\mathrm{r}[(d-2)\vec{\varphi}(\mathrm{r})/2 + \mathrm{r}\nabla_{\mathrm{r}}\varphi(\mathrm{r})]\delta H/\delta\vec{\varphi}(\mathrm{r})$$
$$+ \int_{\mathrm{rr}'} \{h(\mathrm{r}-\mathrm{r}')[\delta^2 H/\delta\vec{\varphi}(\mathrm{r})\delta\vec{\varphi}(\mathrm{r}') - \delta H/\delta\vec{\varphi}(\mathrm{r})\delta H/\delta\vec{\varphi}(\mathrm{r}')]$$
$$-\frac{1}{2}\eta(\mathrm{r}-\mathrm{r}')\vec{\varphi}(\mathrm{r})\delta H/\delta\vec{\varphi}(\mathrm{r}')\}, \qquad (1)$$

where $H = H_{\text{total}} - H_0$, and where H_0 is assumed to be the functional

$$H_0 = \int_{\mathrm{q}} G_0^{-1}(\mathrm{q})|\vec{\varphi}(\mathrm{q})|^2; \qquad \int_{\mathrm{q}} \equiv \int d^{\mathrm{d}}\mathrm{q}/(2\pi)^{\mathrm{d}};$$

$$H = \sum_{\mathrm{k}=1}^{\infty} 2^{1-2\mathrm{k}} \int_{\{q_i, q_{i'}\}} (2\pi)^{\mathrm{d}}\delta\left(\sum_{i=1}^{\mathrm{k}}(\mathrm{q}_i + \mathrm{q}_{i'})\right) \times g_{\mathrm{k}}\{\mathrm{q}_i, \mathrm{q}_{i'}\}\prod_{i=1}^{\mathrm{k}} \vec{\varphi}_{\mathrm{q}_i}\vec{\varphi}_{\mathrm{q}_{i'}}. \qquad (2)$$

Here d is the dimensionality of the space; $\vec{\varphi}$ is an n-component vector; $h(\mathbf{q}) = \exp(-\mathbf{q}^2/2\Lambda^2)$; and the anomalous-dimensionality function $\eta(\mathbf{q})$ is given by

$$\eta(q) = \eta(0) + [(D(q) - D(0)) - \eta(0)G_0^{-1}(q)]/(G_0^{-1}(q) + g_1);$$

$$D(q) = -g_1^2 h(q) + \frac{1}{2}\int_p h(p)\{ng_2(p,-p;q,-q) + g_2(p,-q;q,-p)\}. \quad (3)$$

The quantity $\eta(0) = dD/d(q^2)|_{q=0}$ is the same as the Fisher index η. Formally, at the cutoff momentum $\Lambda\to\infty$ we have $h(q)\to\text{const} = h(0) = 1$ and, correspondingly, $h(\mathbf{r}-\mathbf{r}') = [\Lambda/(2\pi)^{1/2}]^d \exp[-(\mathbf{r}-\mathbf{r}')^2\Lambda^2/2]\to\delta(\mathbf{r}-\mathbf{r}')$. The solution of Eq. (1) can thus be sought in the form of a local functional $H = \Phi_0 = \int_r f(\varphi)(\mathbf{r})$, in a procedure limited to an "ordinary" partial differential equation and $\eta = 0$:

$$\dot{f} = df - \frac{d-2}{2}\varphi\nabla_\varphi f + \nabla_\varphi^2 f - (\nabla_\varphi f)^2. \quad (4)$$

A local equation of this kind was studied in Refs. 14 and 15. It was found that for $2<d<4$ the solution of this equation has a unique physical branch which corresponds to a critical behavior..

At $d\leqslant 3$, an ϵ expansion in the φ^4 model is only a qualitative model of this solution. Asymptotically we have $f(\varphi)\sim\varphi^2/2$ as $\varphi\to\infty$, so a power series in $(\varphi^2)^k$ converges conditionally and cannot be truncated. On the other hand, calculations of the spectrum carried out for this solution for various values $n = 1, 2, 3, \ldots$ yield good critical exponents, implying that $H = \Phi_0$ is a good zeroth approximation. The thrust of the remaining discussion is the idea of discarding the cutoff of the power series in φ in all orders of perturbation theory. This approach makes it possible to avoid an expansion in ϵ at $d\leqslant 3$, but it forces us to use the file $f(\varphi)$, which is known only numerically, as a zeroth approximation.

Equation (1) was, in fact, derived under the normalization $\Lambda\equiv 1$, so the substitution $H = \Phi_0$ leads to the generation of nonlocal corrections because of the term

$$-\int \tilde{h}(\mathbf{r}-\mathbf{r}')\delta H/\delta\vec{\varphi}(\mathbf{r})\delta H/\delta\vec{\varphi}(\mathbf{r}'),$$

where

$$\tilde{h}(\mathbf{r}-\mathbf{r}') = h(\mathbf{r}-\mathbf{r}') - \delta(\mathbf{r}-\mathbf{r}').$$

Retaining the lowest nonvanishing gradients $(\nabla\varphi)^2$, and restricting this letter to the case $n = 1$, we find a generation contribution of the form $\int_r f_{\varphi\varphi}^2(\nabla\varphi)^2/2$. In this case the functional H should be supplemented: $H = \Phi_0 + \Phi_1$, where $\Phi_1 = \int_r \chi(\varphi(r))(\nabla\varphi)^2$. It is not difficult to verify that the function $\eta(r-r')\neq 0$ should be retained in the form $\eta(\mathbf{r}-\mathbf{r}')\approx\eta = \text{const}$ in this order. As a result, we find the following pair of equations for the functions f and χ:

$$\dot{f} = df - \frac{d-2}{2}\varphi f_\varphi + f_{\varphi\varphi} - f_\varphi^2 + B\chi;$$

$$\dot{\chi} = -[\eta + 4f_{\varphi\varphi}]\chi - \left[\frac{d-2+\eta}{2}\varphi + 2f_\varphi\right]\chi_\varphi + \chi_{\varphi\varphi} + \frac{1}{2}[f_{\varphi\varphi}^2 - \eta], \tag{5}$$

where $B = \int_p p^2 h(p) = d/(2\pi)^{d/2} \ll 1$. System of equations (5), like Eq. (4), can be integrated numerically. Since B is numerically small, and since the function χ is expected to be small in comparison with f (this point is verifiable), the last term in the equation for $\dot{f}$ can be omitted in the lowest order in χ. We cannot say this of the parameter $\eta \neq 0$ in the form $(d-2+\eta)/2$ which is to be calculated, since it makes the primary contribution to the physically important term $\propto f_\varphi \varphi$ as $d \to 2$ (Ref. 14). Conditions (3), however, do not determine η completely, since in this order they reduce to the boundary conditions $\chi(0) = \chi_\varphi(0) = 0$. Under these restrictions, and for a fixed branch of the solution, $\dot{f} = 0$, the equation for the fixed point, $\dot{\chi}(\varphi) = 0$, has a one-parameter family of solutions $\chi(\varphi;\eta)$. Again, as in the case with $f(\varphi)$, this circumstance leads to the problem of choosing the physical branch of $\chi(\varphi;\eta)$. This choice, in fact; fixes a value $\eta \neq 0$.

The difficulty, however, is that the physical branch of $\mathbf{f}(\boldsymbol{\varphi})$ is known only numerically, and it varies with η. To get around this difficulty, we make the substitutions $\varphi = \varphi'/(2\Delta_\varphi)^{1/2}$ and $d_{\text{eff}} = 2d/(2-\eta)$ in the equations $\dot{f} = \dot{\chi} = 0$, where $\Delta_\varphi = (d-2+\eta)/2$ is the scale dimensionality of the field φ. We have

$$d_{eff} f/(d_{eff} - 2) - \frac{1}{2}\varphi' f_{\varphi'} + f_{\varphi'\varphi'} - f_{\varphi'}^2 = 0; \tag{6a}$$

$$-[\eta/2\Delta_\varphi + 4f_{\varphi'\varphi'}]\chi - \left[\frac{1}{2}\varphi' + 2f_{\varphi'}\right]\chi_{\varphi'} + \chi_{\varphi'\varphi'} + \frac{1}{2}[f_{\varphi'\varphi'}^2 - \eta] = 0. \tag{6b}$$

Equation (6a) is universal, and the physical branch of $f(\varphi')$ depends unambiguously on d_{eff}. Let us analyze Eq. (6b) now. Noting that at $\varphi \gg 1$ the function $f(\varphi)$ has the asymptotic behavior[14] $f(\varphi) \approx \varphi^2/2 + \ldots$, we find

$$\chi_{\varphi\varphi} - (4+\eta)\chi - (4 + 2\Delta_\varphi)\varphi\chi_\varphi/2 + (1-\eta)/2 = 0,$$

or, at $\varphi \gg 1$ (and $\chi, \chi_\varphi, \chi_{\varphi\varphi} \to \pm\infty$)

$$\chi_{zz} + \left(\frac{1}{2} - z\right)\chi_z - \frac{(4+\eta)}{(4+2\Delta_\varphi)}\chi = 0, \tag{7}$$

where $z = (4 + 2\Delta_\varphi)\varphi^2/4$. Equation (7) is the Kummer equation.[17] It determines the asymptotic behavior $\chi(\varphi) = c(\eta)\varphi^{2\gamma}\exp[(4+2\Delta_\varphi)\varphi^2/4]$, where $\gamma = (4-d-\eta)/[2(2+d+\eta)] > 0$. Under the given conditions $\chi(0) = \chi_\varphi(0) = 0$, the constant c is determined by the parameter η. Note, however, that for either sign of $c \neq 0$, and at sufficiently large $|\varphi|$, the inequality $\chi \gg f$ holds. In other words, these solutions contradict the method by which Eq. (6b) was derived (i.e., the condition $f \gg \chi$ for all φ) and are therefore "extraneous" branches of this equation. The only possibility for satisfying the inequality $f \gg \chi$ is to choose $\chi \to \text{const} = (1-\eta)/2(4+\eta)$ as $|\varphi| \to \infty$. The latter restriction unambiguously determines the Fisher index $\eta = \eta(d_{\text{eff}}) = \eta(d_{\text{eff}}(d))$. Figures 1 and 2 show the physical branches of the functions $f(\varphi)$ and $\chi(\varphi)$, respectively, in the initial part of the range of φ. The inequality $\chi(\varphi) \propto 10^{-2} f(\varphi) \ll f(\varphi)$ is obvious. Table I shows the values of $f(\varphi=0)$, η, and d found numerically for given values of d_{eff}.

At $d \geqslant 3.75$, the numerical methods which we are using become less efficient, but an ϵ expansion is applicable. In the opposite limit, $d \to 2$, the higher gradients become important, and the accuracy of this calculation is again degraded. As a result, the value $\eta \approx 0.28$ turns out to be too high in comparison with the exact result of the Ising model, $\eta(d=2; n=1) = 0.25$.

From this point on the procedure is fairly regular. In the next order of perturbation theory, higher powers of $(\nabla\varphi)$ should be retained, as should the contribution of $B\chi$ to the equation for f. The function $\eta(\mathbf{r}-\mathbf{r}')$ should also be expanded. The coefficient of the first nonvanishing correction is fixed by the condition $\Phi_2 \ll \Phi_{1,0}$ in $H = \Phi_0 + \Phi_1 + \Phi_2 + \dots$.

We conclude with a summary of the basic features of the approach proposed here and the results.

1. The choice of the local functional $\Phi_0 = \int_r f(\varphi(\mathbf{r}))$ as the zeroth renormalization-group approximation leads to a (unique) physical branch of the solution and to

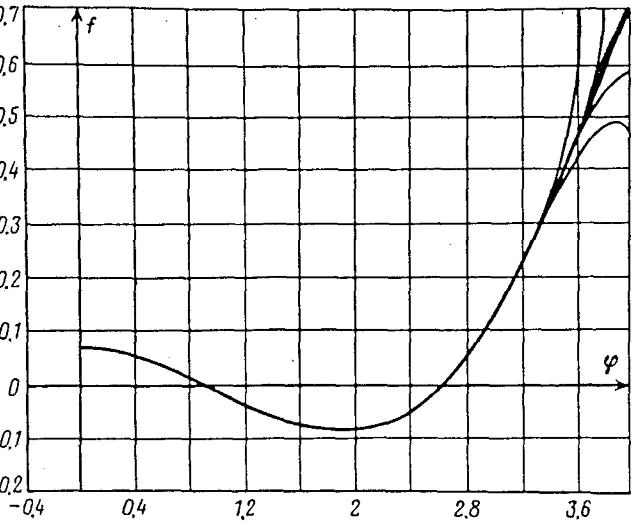

FIG. 1. Physical branch of the local renormalization-group equation (the heavy line) and nearby nonphysical branches, which diverge as $\ln|\varphi-\varphi_0|$ at finite values of φ_0.

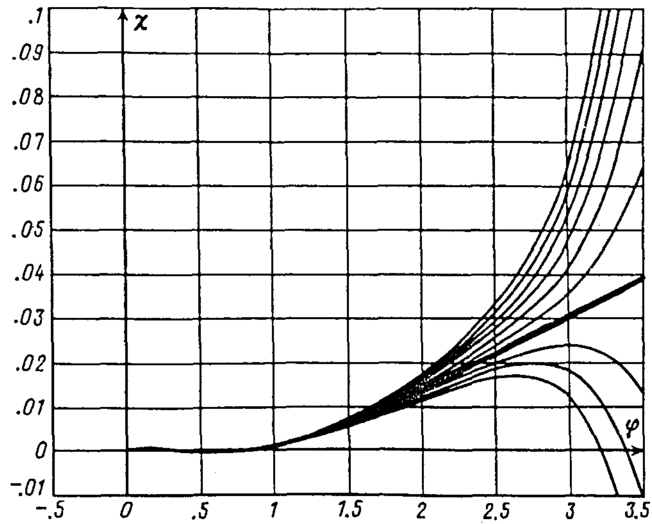

FIG. 2. The branch of $\chi(\varphi)$ which assumes a constant value (the heavy line) and side branches which diverge as $|\varphi| \to \infty$.

good critical exponents. This branch degenerates into the φ^4 model as $d \to 4$. In general, it is only qualitatively similar to the φ^4 model, it has the asymptotic behavior $\sim \varphi^2$, and it requires the retention of all powers of $(\varphi^2)^k$ in the expansion of f in powers of φ.

2. By refraining from cutting off the series in φ in all orders of perturbation theory, we are able to avoid an expansion in ϵ at $d \leqslant 3$, but the entire file $f(\varphi)$ found numerically must be used.

3. Incorporating deviations from a local nature leads to the appearance of corrections to the local-approximation equation, so system of equations (5) must be solved in a self-consistent fashion. Consequently, the quantity η in the combination $(d-2+\eta)/2$ is not comparable to the difference $(d-2)$, which is small as $d \to 2$; on

TABLE I.

d_{eff}	$f(\varphi=0)$	η	d
2,3	0,44(3)	0,279(6)	$\simeq 2$
2,5	0,23(1)	0,15(8)	2,30
2,75	0,13(7)	0,07(5)	2,65
3	0,076(2)	0,03(4)	2,95
3,05	0,067(7)	0,03(1)	$\simeq 3$
3,25	0,041(1)	0,013(4)	3,23
3,5	0,020(0)	0,004(1)	3,49

the contrary, it increases the effective dimensionality d_{eff}, so the expansion works satisfactorily even for $d \to 2$.

4. The small parameter of the theory is actually the quantity $f(0)$, which can be shown to fall off exponentially with increasing d: $f(0) \sim \exp[-a(d-2+\eta)]$, where $a>0$ is on the order of unity. On the other hand, we have $f(0) \sim \epsilon \to 0$ as $d \to 4$. In this limit the efficiency of the numerical methods which we are using is degraded, but an ϵ expansion works. As a result, the approach is most effective for $d \simeq 3$, i.e., just where we would most like it to be.

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