Spontaneous symmetry breaking of the quantum $SL_q(2)$ group

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A mechanism for spontaneous symmetry breaking of q-deformed groups has been proposed. A phenomenological Lagrangian which describes the interaction of Goldstone particles with anomalous statistics has been constructed using the example of the symmetry breaking of the $SL_q(2)$ group.

The discovery of the quantum or q-deformed groups,[1–3] which describe the symmetry of the 2D quantum-field-theory models and the statistical models in the most suitable way, gave rise to the idea of "q-ization" of all of the group-theory structures of the quantum field theory.[4–6] The first such attempt involved q-ization of the Yang–Mills theory.[4,6] In the present letter we will consider the possibility of a spontaneous symmetry breaking of the quantum groups. Using the example of the symmetry breaking of the quantum group $SL_q(2,/R)$, we will construct a phenomenological Lagrangian for the Goldstone fields which have anomalous statistics.

Let us consider the quantum group $SL_q(2,/R)$ of matrices of the type $G=\binom{a\,b}{c\,d}$, which are subject to the condition

$$\det_q G = ad - qbc = 1\ , \tag{1}$$

and which have the elements that satisfy the commutation relations

$$\begin{aligned} &ab = qba, \quad cd = qdc, \qquad bc = cb, \\ &ac = qca, \quad bd = qdb, \quad ad - da = (q - q')bc\ . \end{aligned} \tag{2}$$

To find the exponential mapping of the quantum algebra $sl_q(2)$, it is useful to use instead of the generators H, X^+, and X^-, which satisfy the commutation relations[7,8]

$$[H, X^{\pm}] = 2X^{\pm}, \quad [X^+, X^-] = \frac{\sinh(H \ln q)}{\sinh \ln q}, \tag{3}$$

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the generators $\sigma_{\pm}$, σ_3,[9,10] which are the fundamental representation of $sl_q(2)$ and of the classical $sl(2)$ algebra. The noncommutability of the elements of group (2) determines the noncommutability of the parameters of the algebra.

We will represent the matrix G as a Gaussian expansion

$$G=\begin{pmatrix} (1+qbc)d^{-1} & bd^{-1} \\ cd & d \end{pmatrix}=\begin{pmatrix} 1 & bd^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ cd & 1 \end{pmatrix}\begin{pmatrix} d^{-1} & 0 \\ 0 & d \end{pmatrix}. \tag{4}$$

Each matrix can be represented as an exponential function of a generator $\sigma_{\pm}$, σ_3:

$$G=e^{\varphi_-\sigma_+}e^{\varphi_+\sigma_-}e^{\varphi\sigma_3}=\begin{pmatrix} 1 & \varphi_- \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varphi_+ & 1 \end{pmatrix}\begin{pmatrix} e^{\varphi} & 0 \\ 0 & e^{-\varphi} \end{pmatrix}$$

$$=\begin{pmatrix} \rho+\varphi_-\varphi_+\rho & \varphi_-\rho^{-1} \\ \varphi_+\rho & \rho^{-1} \end{pmatrix}, \tag{5}$$

where

$$\sigma_+=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix};\quad \sigma_-=\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix};\quad \sigma_3=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};\quad \rho=e^{\varphi}.$$

The commutation relations for the parameters of the algebra can be obtained from the direct calculation on the basis of condition (2):

$$\rho\varphi_{\pm}=q\varphi_{\pm}\rho \qquad \varphi_-\varphi_+=q^2\varphi_+\varphi_- . \tag{6}$$

Let us determine the left-invariant differential Cartan forms for $SL_q(2)$[11,12]

$$\Omega=G^{-1}dG=\omega_+\sigma_-+\omega_-\sigma_++\omega_3\sigma_3=\begin{pmatrix} \omega_3 & \omega_- \\ \omega_+ & -\omega_3 \end{pmatrix}, \tag{7}$$

$$\omega_3=\rho^{-1}d\rho+\rho^{-1}d\varphi_-\varphi_+\rho, \tag{8}$$

$$\omega_+=\rho d\varphi_+\rho-\rho\varphi_+d\varphi_-\varphi_+\rho, \tag{9}$$

$$\omega_-=\rho^{-1}d\varphi_-\rho^{-1}. \tag{10}$$

From the requirement that the Maurer–Cartan equations be satisfied we can obtain the commutation relations between the parameters ρ and $\varphi_{\pm}$ and their differentials

$$\begin{aligned} &\rho d\rho=d\rho\rho \qquad \varphi_+d\rho=\frac{1}{q}d\rho\varphi_+ \\ &\rho d\varphi_+=qd\varphi_+\rho \qquad \varphi_+d\varphi_+=d\varphi_+\varphi_+ \\ &\rho d\varphi_-=d\varphi_-\rho \qquad \varphi_+d\varphi_-=\frac{1}{q^2}d\varphi_-\varphi_+, \end{aligned} \tag{11}$$

$$\varphi_- d\varphi_+ = q^2 d\varphi_+ \varphi_- \quad \varphi_- d\rho = \frac{1}{q} d\rho\varphi_- + \left(\frac{1}{q} - 1\right) \rho d\varphi_-$$
$$d\varphi_- d\rho = -d\rho d\varphi_- \qquad \omega_-\omega_+ = -\omega_+\omega_-$$
$$d\varphi_+ d\rho = -\frac{1}{q} d\rho d\varphi_+ \qquad \omega_-\omega_3 = -\omega_3\omega_- \tag{12}$$
$$d\varphi_- d\varphi_+ = -q^2 d\varphi_+ d\varphi_- \qquad \omega_+\omega_3 = -\omega_3\omega_+.$$

The differential d can be expressed as an expansion in ω forms

$$d = \delta\rho \frac{\partial}{\partial\rho} + \delta\varphi_+ \frac{\partial}{\partial\varphi_+} + \delta\varphi_- \frac{\partial}{\partial\varphi_-} = \omega\nabla + \omega_-\nabla_+ + \omega_+\nabla_- = \omega_i\nabla_i, \tag{13}$$

where ∇_i are the covariant derivatives

$$\nabla = \rho\frac{\partial}{\partial\rho} \tag{14}$$

$$\nabla_- = \frac{1}{q}\rho^{-2}\frac{\partial}{\partial\varphi_+} \tag{15}$$

$$\nabla_+ = \rho^2\frac{\partial}{\partial\varphi_-} - \frac{1}{q^2}\rho^3\varphi_+\frac{\partial}{\partial\rho} + \frac{1}{q}\rho^2\varphi_+^2\frac{\partial}{\partial\varphi_+}. \tag{16}$$

The condition $d^2 = 0$ determines the covariant-derivative algebra, which is the same as the $sl(2,/R)$ algebra

$$[\nabla_+, \nabla_-] = \nabla, \quad [\nabla, \nabla_+] = 2\nabla_+, \quad [\nabla, \nabla_-] = 2\nabla_-, \tag{17}$$

and the conditions

$$\frac{\partial}{\partial\varphi_+}\rho - q\rho\frac{\partial}{\partial\varphi_+} = 0 \quad \frac{\partial}{\partial\rho}\frac{\partial}{\partial\varphi_+} - q\frac{\partial}{\partial\varphi_+}\frac{\partial}{\partial\rho} = 0$$
$$\frac{\partial}{\partial\rho}\varphi_+ - \frac{1}{q}\varphi_+\frac{\partial}{\partial\rho} = 0 \quad \frac{\partial}{\partial\rho}\frac{\partial}{\partial\varphi_-} - q\frac{\partial}{\partial\varphi_-}\frac{\partial}{\partial\rho} = 0$$
$$\frac{\partial}{\partial\varphi_-}\rho - q\rho\frac{\partial}{\partial\varphi_-} = 0 \quad \frac{\partial}{\partial\varphi_-}\frac{\partial}{\partial\varphi_+} - q^4\frac{\partial}{\partial\varphi_+}\frac{\partial}{\partial\varphi_-} = 0 \tag{18}$$
$$\frac{\partial}{\partial\varphi_+}\varphi_+ - \varphi_+\frac{\partial}{\partial\varphi_+} = 1.$$

Choosing the diagonal-matrix subgroup as the stationary subgroup H, we will treat the factor-space parameters $G/H - \varphi_-(x)$ and $\varphi_+(x)$ as the Goldstone fields which satisfy the commutation relations (6). Since the parameter $|q| = 1$ for $SL_q(2,/R)$, it can be viewed as a phase multiplier which determines the statistics for the Goldstone fields φ_+ and φ_-: $q = e^{i\gamma}$.

The conditions under which the matrix elements G are self-adjoint lead to the following Hermitian conjugation for the operators $\varphi_{\pm}$, ρ:

$$\rho^{+} = \rho \quad (\varphi_{+})^{+} = \rho^{-1}\varphi_{+}\rho \quad (\varphi_{-})^{+} = \rho\varphi_{-}\rho^{-1}.$$

The phenomenological Lagrangian for such fields can be constructed in a standard way:[13,14]

$$L = \omega_{+}^{\mu}\omega_{-}^{\mu} + \omega_{-}^{\mu}\omega_{+}^{\mu} = \partial_{\mu}\varphi_{+}\partial_{\mu}\varphi_{-} + \partial_{\mu}\varphi_{-}\partial_{\mu}\varphi_{+} - q^{2}\varphi_{+}^{2}(\partial_{\mu}\varphi_{-})^{2} - q^{-2}(\partial_{\mu}\varphi_{-})^{2}\varphi_{+}^{2}, \tag{19}$$

where

$$\partial_{\mu} = \frac{\partial}{\partial x^{\mu}}, \quad \varphi_{\pm} = \varphi_{\pm}(x), \quad \omega_{\pm}(d) = dx_{\mu}\omega_{\pm}^{\mu}.$$

This Lagrangian is Hermitian relative to the Hermitian conjugation operation defined above and has a typical Goldstone structure, except for the fact that the fields $\varphi_{\pm}(x)$ and $\partial_{\mu}\varphi_{\pm}(x)$ are operators which satisfy the q-commutation relations

$$\begin{aligned}
&\varphi_{-}(y)\frac{\partial\varphi_{+}(x)}{\partial x^{\mu}} - q^{2}\frac{\partial\varphi_{+}(x)}{\partial x^{\mu}}\varphi_{-}(y) = 0\\
&\frac{\partial\varphi_{-}(x)}{\partial x^{\mu}}\varphi_{+}(y) - q^{2}\varphi_{+}(y)\frac{\partial\varphi_{-}(x)}{\partial x^{\mu}} = 0\\
&\frac{\partial\varphi_{-}(x)}{\partial x^{\mu}}\frac{\partial\varphi_{+}(y)}{\partial y^{\nu}} - q^{2}\frac{\partial\varphi_{+}(y)}{\partial y^{\nu}}\frac{\partial\varphi_{-}(x)}{\partial x^{\mu}} = 0\\
&\varphi_{-}(x)\varphi_{+}(y) - q^{2}\varphi_{+}(y)\varphi_{-}(x) = 0.
\end{aligned} \tag{20}$$

The commutation relations for the fields with anomalous statistics in the arbitrary space-time dimension will be obtained in the next paper, after a quantization with deformed Poisson's brackets matched with a q-deformed law of multiplication of variables (20).

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