Conductivity critical exponent of exponentially distributed resistances

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The geometric structure of a conducting cluster is determined. The results are used to derive an analytic expression for the conductivity critical exponent of a random network with an exponentially wide spectrum of resistances. The results are compared with the exact bounds and the numerical values available.

The problem of determining the effective conductivity σ^e of a random network with an exponentially wide spectrum of resistances has been the subject of many papers (e.g., Refs. 1–9). One reason for the interest in this problem is that calculations of hopping conductivity frequently reduce to this case.[10]

The simplest formulation of the problem of determining σ^e is as follows: The resistance of bond i of the lattice is $R_i = R_0 \exp(-\lambda x_i)$, where R_0 is a constant, and x_i is a random variable with a smooth probability density $D(x_i)$. There are no correlations between the values of different bonds, and the distribution $D(x)$ is the same for all bonds. For simplicity we set $D(x) = 1$ for $0 \leqslant x \leqslant 1$ and $D(x) = 0$ otherwise. In this case, an "exponentially large inhomogeneity" is understood as the limit $\lambda \to \infty$.

Tyc and Halperin[8] recently proposed the following asymptotic expression for the effective conductivity:

$$\sigma^e \approx \frac{C}{R(x_c) a_0^{d-2}} \left(\frac{D(x_c)}{\lambda} \right)^y , \qquad (1)$$

where C is a constant, a_0 is the lattice constant, x_c is defined in terms of the percolation threshold p_c,

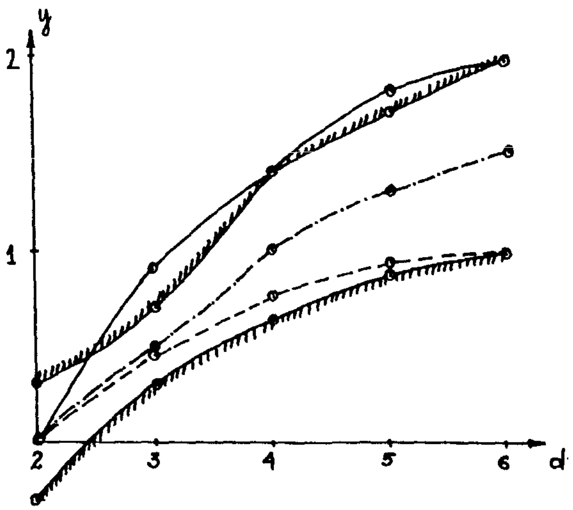

FIG. 1. The critical exponent y versus the dimension of the problem, d. Lines with hatching—Upper and lower bounds;[8,10] solid line—le Doussal's critical exponent;[9] dashed lines—$y_{\rm WLM} = (t+q)/(t-q)$; dot-dashed line—$\tilde{y} - (t-q)/2$. The numerical values of the critical exponents were assumed to be $\nu_2 = 4/3$, $t_2 = q_2 = 1.29$; $\nu_3 = 0.9$, $t_3 = 1.7$, $q_3 = 0.7$; $\nu_4 = 0.7$, $t_4 = 2.4$, $q_4 = 0.35$; $\nu_5 = 0.6$, $t_5 = 2.7$, $q_5 = 0.14$; and $\nu_6 = 0.5$, $t_6 = 3$, $q_6 = 0$. The subscripts here specify the dimension of the problem. The dependence of y on d is shown by solid lines for clarity.

$$\int_{x_c}^{1} D(x)dx = p_c, \tag{2}$$

and y is a critical exponent. The determination of this exponent was the goal of Refs. 8 and 9, and it is also the goal of the present letter.

The analytic expression derived for y in Ref. 9 [see expression (11) below and Fig. 1] is at odds with the numerical values and two-sided bounds found in Refs. 8 and 10. In the present letter we offer an approach which resolves this contradiction.

To determine the critical exponent y we first assume that in the limit $x \to x_c$ we are above the percolation threshold. In other words, if we color all the resistances with $R \leqslant R(x_c)$ black, we find a standard nodes-links-blobs pattern with a certain length of the bridge (of the set of singly connected links). The resistance of this bridge determines the entire resistance in the correlation volume in the case of a large inhomogeneity. Using the expression for $D(x)$ and assuming $\lambda \gg 1$, we find the average resistance of the bridge to be

$$< R >_1 = \int_{x_c}^{1} R(x)P_1(x)dx \approx \frac{R(x_c)}{1-x_c}\lambda^{-1}, \tag{3}$$

where the renormalization of the distribution, $P_1(x) = D(x)/\int_{x_c}^{1} D(x)dx$, is carried out to satisfy the condition that the largest resistance in the bridge be $R(x_c)$.

The resistance of the bridge and therefore the resistance of the entire correlation volume is

$$R_1 = N_1 < R >_1, \tag{4}$$

where N_1 is the number of links in the bridge.

At this point we assume that the medium is below the percolation threshold as $x \to x_c$. Consequently, if we color the resistances with $R \leqslant R(x_c)$ black, we obtain the dual of the picture in the preceding case: All the resistance is lumped in a thin interlayer between well-conducting (black) clusters. The width of this layer is one bond (a_0). The average resistance of this layer is (the bonds in the layer are parallel to each other)

$$< \frac{1}{R} >_2 = \int_0^{x_c} \frac{P_2(x)}{R(x)} dx \approx \frac{\lambda^{-1}}{x_c R(x_c)}, \tag{5}$$

where the renormalization of the distribution, $P_2(x) = D(x)/\int_0^{x_c} D(x)dx$, is carried out because the conducting path includes the smallest resistance in the layer.

The resistance of the layer and thus the resistance of the entire correlation volume are

$$R_2 = 1/N_2 < 1/R >_2 . \tag{6}$$

The quantity N_i is a power function of $\tau = (n(x_c) - n(x))/n(x_c)$, i.e., of the distance from p_c, where $n(x) = \int_{x_c}^1 D(x)dx$:

$$N_i \propto |\tau|^{-\alpha_i}. \tag{7}$$

If x is sufficiently close to x_c, the system is in a smearing region:[6] $\tau < \Delta$, where Δ is the smearing region. In this case we have $R_1(\tau \approx \Delta) = R_2(\tau \approx \Delta)$ and thus

$$\Delta \approx [x_c(1 - x_c)]^{-\frac{1}{\alpha_1 + \alpha_2}} \lambda^{-\frac{2}{\alpha_1 + \alpha_2}}. \tag{8}$$

To determine σ^e in the smearing region, we now use $\sigma^e = L^{d-2}/R_i$, where $L = a_0 |\tau|^{-\nu}$ is a length scale of the correlation volume, and ν is the critical exponent of the correlation volume. Setting $\tau \approx \Delta$ [see Eq. (8)] in σ^e, we find

$$\sigma^e = \frac{1}{a_0^{d-2} R(x_c)} x_c^{-\frac{\alpha_1 + \nu(d-2)}{\alpha_1 + \alpha_2}} (1 - x_c)^{\frac{\alpha_2 + \nu(d-2)}{\alpha_1 + \alpha_2}} \lambda^y, \tag{9}$$

where the critical exponent y is

$$y = \frac{\alpha_1 - \alpha_2 + 2\nu(d-2)}{\alpha_1 + \alpha_2}. \tag{10}$$

The values of α_i are determined in different ways in different models of percolation clusters. In the nodes-links-blobs model, for example, the exponent α_1, which determines the number of singly connected bonds,[12–15] is one. In the model which is the dual of the nodes-links-bonds model, the index α_2 determines the number of singly disconnecting bonds and is also one, according to Ref. 16. According to the nodes-links-blobs models we thus have $\alpha_1 = \alpha_2 = 1$, and the expression derived above for the critical exponent becomes

$$y_{LD} = \nu(d-2), \tag{11}$$

in agreement with le Doussal's result.[9]

According to the weak-link model (WLM), we have[17-20]

$$\alpha_1 = t - \nu(d-2), \quad \alpha_2 = q + \nu(d-2) \tag{12}$$

and thus

$$y_{WLM} = \frac{t-q}{t+q},$$

where t and q are the conductivity critical exponents of two-phase systems.[12] (In two-phase systems with $\sigma_2 \ll \sigma_1$, we have $\sigma^e \propto \tau^t$ at $\tau > 0$ and $\sigma^e \propto |\tau|^{-q}$ at $\tau < 0$.)

The critical exponent y_{WLM} (in contrast with y_{LD}; Ref. 9) agrees well with the values found through a numerical simulation by Tyc and Halperin.[9] It satisfies the two-sided bounds established in Refs. 10 and 9. A more detailed and more involved calculation, which requires the construction of a weak-link model directly in the smearing region, leads to $\tilde{y} = (t-q)/2$.

In summary, the weak-link model, which describes the structure of a two-phase medium near p_c, makes it possible to express the critical exponent y in terms of the well-known critical exponents t and q. This extension of the weak-link model to systems with an exponentially wide spectrum of resistances allows one to apply this model to research on the critical behavior of the $1/f$ noise and of the higher moments of the current distribution in such systems, as was done in Refs. 18 and 20 for two-phase systems. Corresponding results will be published separately.

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[1]V. Ambegaokar, B. I. Halperin, and J. S. Langer, Phys. Rev. B **4**, 2612 (1971).
[2]B. I. Shklovskiĭ and A. L. Efros, Zh. Eksp. Teor. Fiz. **60**, 867 (1971) [Sov. Phys. JETP **33**, 4681 (1971)].
[3]D. Stauffer, *Introduction to Percolation Theory*, Taylor and Francis, London, 1985.
[4]V. Ambegaokar, S. Cochran, and J. Kurkijarvi, Phys. Rev. B **8**, 3682 (1973).
[5]M. Pollak and B. I. Halperin, Solid State Commun. **13**, 869 (1973).
[6]B. I. Shklovskiĭ and A. L. Efros, Usp. Fiz. Nauk **117**, 401 (1975) [Sov. Phys. Usp. **18**, 845 (1975)].
[7]J. Kurkijarvi, Phys. Rev. B **9**, 770 (1974).
[8]S. Tyc and B. I. Halperin, Phys. Rev. B **39**, 877 (1989).
[9]P. le Doussal, Phys. Rev. B **39**, 881 (1989).
[10]E. Charlaix, E. Gudon, and S. Roux, in *Transport in Porous Media*, 2, Riedel, Dordrecht, 1987, p. 31.
[11]B. I. Shklovskiĭ and A. L. Efros, *Electronic Properties of Doped Semiconductors*, Springer-Verlag, Berlin, 1983.
[12]A. L. Efros and B. I. Shklovskiĭ, Phys. Status Solidi B **76**, 475 (1976).
[13]A. S. Skal and B. I. Shklovskiĭ, Fiz. Tekh. Poluprovodn. **8**, 1586 (1974) [Sov. Phys. Semicond. **8**, 1029 (1975)].
[14]A. Coniglio, Phys. Rev. Lett. **46**, 250 (1981).

[15]R. Pike and K. E. Stanley, J. Phys. A **14**, L169 (1981).
[16]A. Coniglio, J. Phys. A **15**, 3829 (1982).
[17]D. C. Wright, D. J. Bergman, and Y. Kantor, Phys. Rev. B **33**, 396 (1986).
[18]A. A. Snarskiĭ, Zh. Eksp. Teor. Fiz. **91**, 1405 (1986) [Sov. Phys. JETP **64**, 828 (1986)].
[19]A. E. Morozovskiĭ and A. A. Snarskiĭ, Zh. Eksp. Teor. Fiz. **95**, 1844 (1989) [Sov. Phys. JETP **68**, 1066 (1989)].
[20]A. E. Morozovskiĭ and A. A. Snarskiĭ, Pis'ma Zh. Eksp. Teor. Fiz. **52**, 871 (1990) [JETP Lett. **52**, 244 (1990)].
[21]A. E. Morozovskiĭ and A. A. Snarskiĭ, Zh. Eksp. Teor. Fiz. **102**, 683 (1992) [Sov. Phys. JETP **75**, 366 (1992)].

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