Dynamics of the boundaries of the oscillation region in a dissipationless shock wave

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Exact (algebraic) expressions describing the dynamics of the trailing and leading edges of the oscillation region in a dissipationless shock wave are derived on the basis of the Gurevich–Pitaevskiĭ hypothesis.

1. We consider the Korteweg–de Vries equation

$$u_t+uu_x=u_{xxx}=0, \quad u=u(x,t), \tag{1}$$

with a monotonically decreasing initial profile: $u(x,0)=u_0(x)$, $u_0(x)=r_0^-(x)>0$ for $x<0$, $u_0(0)=0$; and $u_0(x)=r_0^+(x)<0$ for $x>0$, $u_{0x}<0$, $|u_{0xxx}|\ll|u_0u_{0x}|$ as $|x|\to\infty$. The Gurevich–Pitaevskiĭ hypothesis[1] is that the asymptotic form of the solution of (1) consists of "external" and "internal" solutions. The external solution satisfies the Hopf equation ($u_t+uu_x=0$) and has the solution (we are introducing the change of notation ($u\to r^{\mp}$) $\psi^-(r^-)=x-r^-t$ for $x\leqslant x^-(t)$, $\psi^+(r^+)=x-r^+t$ for $x\geqslant x^+(t)$, where $\psi^{\mp}(r_0^{\mp})=x$ is determined by the initial profile, and $x^-(t)$ and $x^+(t)$ are the boundaries of the trailing and leading edges of the oscillation region. The internal solution is $u=\varphi(\theta)=\varphi(\theta+1)$, where

$$\varphi=2(r_3-r_1)dn^2\,[2K(m)\theta;m]+r_1+r_2-r_3, \quad x^-(t)\leqslant x\leqslant x^+(t), \tag{2}$$

where $\theta_x=\kappa$, $\theta_t=w=-\kappa U$, $U=(r_1+r_2+r_3)/3$, $\kappa\cong(r_3-r_1)^{1/2}/K(m)$, $m=(r_2-r_1)/(r_3-r_1)$, dn is the Jacobi elliptic function, *K(m)* is the complete elliptic integral of the first kind, ($r_3\geqslant r_2\geqslant r_1$) and the r_i satisfy the Whitham equations[1–4]

$$r_t^i+(Q_iU)r_x^i=0, \quad (i=1,2,3),$$

$$Q_i=1+[\kappa/(D_i\kappa)]D_i, \quad D_i\equiv d/dr_i, \tag{3}$$

with the boundary conditions

$$x=x^-(t):\quad r_2=r_1\equiv r_1^-, \quad r_3\equiv r_3^-=r^-,$$

$$x=x^+(t):\quad r_2=r_3\equiv r_3^+, \quad r_1\equiv r_1^+=r^+. \tag{4}$$

These boundary conditions keep the multivalued function $\{r^{\mp},r_i\}$ continuous at the boundaries of the region. An important point[3] is that the plot of the function $\{r^{\mp},r_i\}$ must be C^1-smooth, even at the boundaries of the oscillation region.

A solution of (3), (4) is given by the generalized hodograph method:[5]

$$\psi_i(r)=x+t\varphi_i(r), \tag{5}$$

where[4] $\psi_i = Q_i V(r)$, and V satisfies the system[4]

$$2(r_j - r_i) D_i D_j V + D_i V - D_j V = 0, \quad i \neq j, \tag{6}$$

with the boundary conditions[6]

$$V^- + 2(r_3^- - r_1^-) D_3 V^- = \psi(r_3^-), \quad V^+ - 2(r_3^+ - r_1^+) D_1 V^+ = \psi(r_1^+), \tag{7}$$

where $V^- = V(r_1, r_1, r_3)$, $V^+ = V(r_1, r_3, r_3)$, and $\psi(u_0) = x$. Problem (6), (7) was first solved in Ref. 6. Its solution can conveniently be written in the form[7]

$$V(r^1, r^2, r^3) = \frac{1}{2\pi} \int_{r^2}^{r^3} \int_{r^1}^{r^2} \frac{\psi(\beta) d\lambda d\beta}{R(\lambda, \beta)} + \frac{1}{2\pi} \int_{r^1}^{r^2} \int_{r^2}^{r^3} \frac{\psi(\beta) d\lambda d\beta}{R(\lambda, \beta)},$$

where $R(\beta, \lambda) = \sqrt{(\beta - \lambda) \Pi_1^3 (\lambda - r_i)}$.

Our purpose in the present letter is to analyze the behavior of solution (5) near the boundaries of the oscillation region, $x^-(t)$ and $x^+(t)$. In other words, we wish to find exact (algebraic) equations which describe the dynamics of the trailing and leading edges and to prove the C^1 smoothness of the plot of solution (5) near $x^-(t)$ and $x^+(t)$.

2. We introduce $(V_i^-, V_{ik,...}^-) \equiv (D_i V, D_i D_k V, ...) |_{r_2 = r_1}$ and $(V_i^+, V_{ik}^+, ...) \equiv (D_i V, D_i D_k V, ...) |_{r_2 = r_3}$, where V is the solution of (6), (7). We assume $V_1^- = V_2^-$, $V_{11}^- = V_{22}^- = 3V_{12}^-$, $V_{111}^- = V_{222}^- = 5V_{122}^- = 5V_{211}^-$ and $V_2^+ = V_3^+$, $V_{22}^+ = V_{33}^+ = 3V_{23}^+$. We note that if (for example) the function $\psi(\beta)$ can be written as a sum over odd powers of β: $\psi(\beta) = \Sigma C_n \beta^{2n+1}$ ($n = 0, 1, 2, ...$), then solution (6), (7) is a sum of polynomials which are symmetric under all possible interchanges $(r_i \to r_j, r_j \to r_i)$. The relations written above are thus satisfied automatically.

We assume that V satisfies these relations. The asymptotic form of solution (5) is thus as follows: near the trailing boundary, $x = x^-(t) + x'$,

$$r_1 = r_1^-(t) - a + ...,$$

$$r_2 = r_1^-(t) + a + ...,$$

$$r_3 = r_3^-(t) + b + ..., \tag{8}$$

where

$$a = (-2 r_{1t}^- x')^{1/2}, \tag{9}$$

$$b = x' / [t + \partial \psi^-(r_3^-) / \partial r_3^-], \tag{10}$$

and $\{x^-(t), r_1^-(t), r_3^-(t)\}$ are given by the equations

$$x^-(t) = \psi^-[r_3^-(t)] + t r_3^-(t),$$

$$2V_1^- + V_3^- + t = 0,$$

$$3V_{31}^- + 4V_{11}^- = 0; \tag{11}$$

near the leading boundary, $x = x^+(t) - x''$.

$$r_1 = r_1^+(t) + B + \dots,$$

$$r_2 = r_3^+(t) - A + \dots,$$

$$r_3 = r_3^+(t) + A + \dots,$$

where

$$A^2[\ln(8(r_3^+ - r_1^+)/A) + 1/2] = -3x'' r_{3t}^+, \tag{13}$$

$$B = -x''/[t + \partial\psi^+(r_1^+)/\partial r_1^+] + 0[x'' \ln(x'')], \tag{14}$$

and $\{x^+(t), r_1^+(t), r_3^+(t)\}$ are determined by the equations

$$x^+(t) = \psi^+[r_1^+(t)] + t r_1^+(t),$$

$$3V_1^+ + t = 0,$$

$$V_{13}^+ = 0. \tag{15}$$

The dynamics of the trailing edge and the leading edge of the oscillation region in a dissipationless shock wave is thus described by Eqs. (8)–(15). It follows from (8)–(10) and (12)–(14) that the plot of the multivalued function $\{r^{\mp}, r_i\}$ is C^1-smooth near $x^-(t)$ and $x^+(t)$.

Equations (9) and (13) were derived previously by Gurevich and Pitaevskiĭ[1] (see also Ref. 3). In the case $\psi(\beta) = -\beta^3$, Eqs. (8)–(15) correspond to the result of Ref. 8.

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