Evolution of a nonmonotonic perturbation in Korteweg–de Vries hydrodynamics

A. L. Krylov,[1)] V. V. Khodorovskiĭ,[1)] and G. A. Él'

Institute of Terrestrial Magnetism, the Ionosphere, and Radio Wave Propagation, Russian Academy of Sciences, 142092, Troitsk, Moscow Oblast, Russia;[1)]O. Yu. Shmidt Institute of Earth Physics, Russian Academy of Sciences, 123810, Moscow, Russia

(Submitted 29 June 1992; resubmitted 12 August 1992)
Pis'ma Zh. Eksp. Teor. Fiz. **56**, No. 6, 325–329 (25 September 1992)

A general solution is derived for the evolution of a large-scale nonmonotonic perturbation in dispersive Korteweg–de Vries hydrodynamics.

1. The problem of a shock wave in dispersive hydrodynamics was first formulated by Gurevich and Pitaevskiĭ. They used Whitham's method[2] to find a self-similar solution of the problem of the decay of an initial discontinuity in Korteweg–de Vries (KdV) hydrodynamics. The construction of solutions of a more general type became possible as the result of a generalization of the classical hodograph method to the multidimensional case [Ref. 3 (see also Ref. 4); "multidimensional" here refers to the space of dependent variables]. Solutions of the Gurevich–Pitaevskiĭ problem were found in Refs. 5–7 for the KdV equation with initial conditions corresponding to (a) a monotonic breaking profile and (b) a localized perturbation. In the latter case, a wave moving through the unperturbed medium was analyzed. The solutions for both the monotonic and localized cases were characterized by two arbitrary functions specifying the initial profile.

In the present letter we construct a solution of the Gurevich–Pitaevskiĭ problem for the KdV equation with a slight dispersion,

$$\partial_t u + u\partial_x u + \epsilon^2 \partial^3_{xxx} u = 0, \quad \epsilon^2 \ll 1, \tag{1}$$

with initial conditions of a general type (Fig. 1):

$$u(0,x) = u_0(x) = \begin{cases} r_0^+(x), & x \geqslant 0, \\ r_0^-(x), & x^* \leqslant x < 0, \\ r_0^{\mathrm{II}}(x), & x \leqslant x^*, \end{cases} \tag{2}$$

where $r_0^+(0) = r_0^-(0) = 0$; $(r_0^+)'$, $(r_0^-)' \leqslant 0$; $(r_0^{\mathrm{II}})' \geqslant 0$; $r_0^{\mathrm{II}}(-\infty) \geqslant r_0^+(+\infty)$; $r_0^-(x^*) = r_0^{\mathrm{II}}(x^*) = h$; $u_0'(0) \to -\infty$. The function $u(0,x)$ is generally nonanalytic at the (unique) breaking point (0,0) and also at the point of the maximum (x^*,h). In addition, $u_0(x)$ varies slowly; i.e., we have $u_0/(u_0') \gtrsim 1$ everywhere except in an ϵ-neighborhood of the breaking point.

The breaking of profile (2) in dispersive hydrodynamics (1) is known[1] to lead to the formation of a dissipationless shock wave: a quasisteady oscillating region which lies between the leading edge $x^+(t)$ and the trailing edge $x^-(t)$. These boundaries are

0021-3640/92/000323-05$03.00

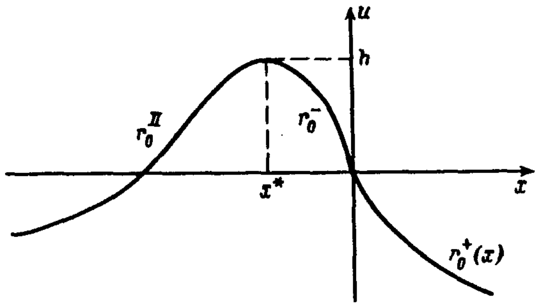

FIG. 1. The initial perturbation $u=u_0(x)$.

unknown at the outset. In the case of localized initial conditions, the dissipationless shock wave converts asymptotically ($t\to\infty$) into a soliton wave: a chain of noninteracting solitons.[7]

The dynamics of the dissipationless shock wave is described by Whitham's modulation system, found by averaging the initial equation over the period of the steady-state traveling wave. In the KdV case, the modulation system is of Riemannian form:

$$\partial_t r_i + V_i(r)\partial_x r_i = 0, \quad i=1,2,3, \tag{3}$$

where the characteristic velocities can be specified in a potential representation,[4]

$$V_i(r) = U - [3\partial_i \ln(\lambda/\epsilon)]^{-1}. \tag{4}$$

Here

$$U = \frac{1}{3}\sum_{i=1}^{3} r_i \text{ and } \lambda = 2\times 6^{1/2}\epsilon \int_{r_i}^{r_2}\left[\prod_{i=1}^{3}(\tau - r_i)\right]^{-1/2} d\tau$$

are the phase velocity and length of the wave, and $r_3 \geqslant r_2 \geqslant r_1$. The solution of Eqs. (3) is joined at the boundaries of the dissipationless shock wave with the smooth solution $r(x,t)$ of the Hopf equation

$$\partial_t r + r\partial_x r = 0, \tag{5}$$

with the initial conditions $r(0,x) = u(0,x)$, since the dispersive term in KdV equation (1) is unimportant.

The boundary conditions on system (3), (4) (the Gurevich–Pitaevskiĭ conditions) are[1]

$$\begin{aligned} &r_2(x^-,t) = r_1(x^-,t), \quad r_3(x^-,t) = r(x^-,t), \\ &r_2(x^+,t) = r_3(x^+,t), \quad r_1(x^+,t) = r(x^+,t). \end{aligned} \tag{6}$$

2. A solution of system (3) can be found in the form

$$x - V_i(r)t = W_i(r), \quad r \equiv (r_1, r_2, r_3), \tag{7}$$

where

$$W_i = f - \partial_i f / \partial_i \ln(\lambda/\epsilon), \tag{8}$$

and the function $f(r)$ (a scalar potential) satisfies the Eisenhart system[8] (see Refs. 5 and 6 for a detailed derivation):

$$\partial^2_{ij} f = \frac{1}{2(r_i - r_j)} (\partial_i f - \partial_j f). \tag{9}$$

System (9) was first derived for the KdV case by a direct calculation in Ref. 9.

It is clear from (7) that if the initial profile, (2), is nonmonotonic, then the generalized hodograph transformation (7) is of a "two-sheet" nature. The solution of the problem, (3), (6), is thus given by two systems (7)–(9) with two different functions $f_{\rm I,II}$ (which are defined in the region $r_1 \leqslant r_2 \leqslant r_3 \leqslant h$). The joining of these functions occurs along the characteristic plane $r_3 = h$. For these functions we have the boundary conditions[5-7]

$$\begin{aligned} f_{\rm I}(r_1,0,0) &= \frac{1}{2(-r_1)^{1/2}} \int_0^{-r_1} y^{-1/2} W_{\rm I}^+(-y)\,dy, \\ f_{\rm I}(0,0,r_3) &= \frac{1}{2r_3^{1/2}} \int_0^{r_3} y^{-1/2} W_{\rm I}^-(y)\,dy, \end{aligned} \tag{10}$$

where $f_{\rm I}(r_1,r_1,0)$ and $f_{\rm I}(0,r_3,r_3)$ are bounded;

$$f_{\rm II}(0,0,r_3) = \begin{cases} \dfrac{1}{2r_s^{1/2}} \displaystyle\int_0^{r_s} y^{-1/2} W_{\rm II}^-(y)\,dy, & r_3 > 0, \\ \dfrac{1}{2(-r_3)^{1/2}} \displaystyle\int_0^{-r_3} y^{-1/2} W_{\rm II}^-(-y)\,dy, & r_3 < 0, \end{cases} \tag{11}$$

$$f_{\rm I}(r_1,r_2,h) = f_{\rm II}(r_1,r_2,h). \tag{12}$$

Here the functions $x = W_{\rm I}^+(r)$, $x = W_{\rm I}^-(r)$, and $x = W_{\rm II}^-(r)$, are the inverses of $r = r_0^+(x)$, $r = r_0^-(x)$, and $r = r_0^{\rm II}(x)$, respectively.

3. The solution $f_{\rm I}(r)$ on the first sheet is the same as that found in Ref. 6 for the problem of the breaking of a nonanalytic monotonic profile:

$$f_{\rm I}(r) = \frac{1}{\pi(r_3 - r_2)^{1/2}} \int_{r_2}^{r_3} \frac{W_{\rm I}(y)}{(y - r_1)^{1/2}} K(z)\,dy + \frac{1}{\pi(r_2 - r_1)^{1/2}} \int_{r_1}^{r_2} \frac{W_{\rm I}(y)}{(r_3 - y)^{1/2}} K(z^{-1})\,dy. \tag{13}$$

Here

$$W_{\rm I}(y) = \begin{cases} W_{\rm I}^+(y), & y \leqslant 0 \\ W_{\rm I}^-(y), & y > 0; \end{cases} \quad z = \left[\frac{(r_2 - r_1)(r_3 - y)}{(r_3 - r_2)(y - r_1)} \right]^{1/2}, \tag{14}$$

and $K(z)$ is the complete elliptic integral of the first kind—the Riemann function of the Euler–Poisson equation.[10]

The solution on the second sheet, which satisfies conditions (11) and (12), is

$$f_{\rm II}(r)=f_{\rm I}(r)+\frac{1}{\pi(r_3-r_1)^{1/2}}\int_{r_3}^{h}\frac{D(y)}{(y-r_2)^{1/2}}K(z_1)dy, \tag{15}$$

where

$$D(y)=W_{\rm I}(y)-W_{\rm II}(y),\quad D(h)=0,\quad z_1=\left[\frac{r_2-r_1}{r_3-r_1}\,\frac{y-r_3}{y-r_2}\right]^{1/2}.$$

The continuity of solution (7), (8), (14), (15) in the region where the sheets join is ensured by the continuity of the normal derivative $\partial_3 f(r_1,r_2,h)$. If $r_1\equiv 0$, solution (15), (16) is the same as the solution of the problem of a quasisimple wave.[7]

4. Let us examine the evolution of a localized perturbation $r_0^{+}(+\infty)\to r_0$, $r_0^{\rm II}(-\infty)\to r_0$, with $r_0\leqslant 0$ const. We know[7] that in the limit $t\to\infty$ the solution is a soliton wave in which the relation $r_2=r_3$ holds, and the solitons are moving against a homogeneous background r_0.

As $r_2\to r_3$, solution (7), (8), (15) becomes

$$x\simeq\frac{2r_3+r_0}{3}\,t+x_0(r_3), \tag{16}$$

$$t\simeq T(r_3)\ln(1-m),\quad m=\frac{r_2-r_0}{r_3-r_0}. \tag{17}$$

Here

$$x_0(r_3)=\frac{1}{2\pi(r_3-r_0)^{1/2}}\int_{r_3}^{h}\frac{D(y)}{(y-r_3)^{1/2}}\ln\left(\frac{y-r_3}{y-r_0}\right)dy$$

$$+\frac{1}{2(r_3-r_0)^{1/2}}\int_{r_0}^{r_3}\frac{W_{\rm I}(y)}{(r_3-y)^{1/2}}\,dy, \tag{18}$$

$$T(r_3)=\frac{3}{4\pi(r_3-r_0)^{1/2}}\int_{r_3}^{h}\frac{D'(y)}{(y-r_3)^{1/2}}\,dy. \tag{19}$$

We see that we have $T(r_3)<0$, as we must have for $t\to+\infty$.

Equations (17) and (19) can be used to calculate the density of solitons in the soliton wave:

$$k=\frac{2\pi}{\lambda}=\frac{\pi(r_3-r_1)^{1/2}}{\epsilon 6^{1/2}K(m)}\xrightarrow{r_2\to r_3}-\frac{2\pi(r_3-r_1)}{\epsilon 6^{1/2}\ln(1-m)}. \tag{20}$$

Comparing with (17), we find the following result for the soliton wave:

$$\epsilon k\simeq\frac{1}{t}\left[-\frac{2\pi}{6^{1/2}}(r_3-r_0)^{1/2}T(r_3)\right]. \tag{21}$$

This result agrees with the result found by Karpman,[11] which was derived by the method of the inverse scattering problem in the semiclassical limit. Karpman's formula cannot be used to unambiguously reconstruct the initial perturbation from the parameters of the soliton wave, since the function $T(r_3)$ in (19) contains only the width of the initial profile, $D(y)$. A single-valued relationship between the initial data and the soliton wave can be established with the help of (16)–(19).

We wish to thank A. V. Gurevich for his time and interest.

[1] A. V. Gurevich, Zh. Eksp. Teor. Fiz. **65**, 590 (1973) [Sov. Phys. JETP **38**, 291 (1973)].
[2] G. B. Whitham, Proc. R. Soc. A **283**, 238 (1965).
[3] S. P. Tsarev, Dokl. Akad. Nauk SSSR **282**, 534 (1985).
[4] B. A. Dubrovich and S. P. Novikov, Usp. Mat. Nauk **44**, 29 (1989).
[5] A. V. Gurevich, A. L. Krylov, and G. A. Él', Pis'ma Zh. Eksp. Teor. Fiz. **54**, 104 (1991) [JETP Lett. **54**, 102 (1991)].
[6] A. V. Gurevich, A. L. Krylov, and G. A. Él', Zh. Eksp. Teor. Fiz. **101**, 1797 (1992) [Sov. Phys. JETP **74**, 957 (1992)].
[7] A. V. Gurevich, A. L. Krylov, N. G. Mazur, and G. A. Él', Dokl. Akad. Nauk SSSR **323**, 876 (1992).
[8] L. P. Eisenhart, Annals of Math. **120**, 262 (1918).
[9] V. R. Kudashev and S. E. Sharapov, Teor. Mat. Fiz. **87**, 40 (1991).
[10] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Interscience, New York, 1962.
[11] V. I. Karpman, Phys. Lett. A **25**, 708 (1967).

Translated by D. Parsons