Exact solution for the SU(N) main chiral field in two dimensions

P. B. Wiegmann

Institute of Theoretical Physics, Academy of Sciences of the USSR

(Submitted 4 January 1984)

Pis'ma Zh. Eksp. Teor. Fiz. **39**, No. 4, 180–183 (25 February 1984)

An exact solution is constructed for the SU(N) main chiral field in two dimensions. The particle spectrum, the S matrix, and the expectation values of the chiral currents in external fields are determined from this solution.

1. The low-energy part of the interaction of Goldstone particles, which appear upon spontaneous symmetry breaking, is determined completely by the symmetry group G and is described by a $G \otimes G$-invariant nonlinear σ model:

$$S = \frac{1}{2\lambda_0} \int d^2x \, \mathrm{tr}\,(g^{-1}\partial_\mu g)^2 , \tag{1}$$

where $g(x)$, the so-called main chiral field, is an element of group G.

We know that in two dimensions (2D) the effective interaction of non-Abelian Goldstone fields increases without bound with decreasing energy scale.[1] Consequently, the low-energy properties of the system, in particular, the spectrum of the particles, lie outside the range of applicability of the standard methods of quantum field theory. We know, on the other hand, that a 2D chiral field has an infinite series of conservation laws[2,3] and a factorized scattering theory, so that it is completely integrable.

In this letter we summarize the results of an exact solution of the SU(N) main chiral field based on an idea suggested by Polyakov several years ago (see Ref. 4, for example). In particular, we show that non-Abelian Goldstone bosons are massive and form the basis of a ring of representations of the group $G \otimes G$ with the typical spectrum for $G = \mathrm{SU}(N)$:

$$m_k = m \frac{\sin((\pi k)/N)}{\sin(\pi/N)} ; \qquad k = 1 \,....\, N-1. \tag{2}$$

2. Our method is described in detail in Ref. 5, where a solution is derived for the simplest SU(2) chiral model. The method is based on the equivalence of the chiral field and the (1 + 1) model of interacting fermions. The latter is also integrable, but it has been traditional to apply the Bethe Ansatz to it.

We denote by $\psi \equiv \psi_f^\alpha (\alpha = 1...N; f = 1...N_f)$ the fermion field which forms the "colored" multiplet and the auxiliary "flavored" multiplet; $j_\mu^a \equiv \Sigma_f \bar{\psi}_f \gamma_\mu \tau^a \psi_f$; the τ^a are the generators of SU(N). It is shown in Ref. 5 that in the limit $N_f \to \infty$ a model with the Lagrangian

$$\mathcal{L} = i\bar{\psi}\hat{\partial}\psi + \lambda_0 j_\mu^a j_a^\mu \tag{3}$$

is equivalent to the main chiral field.

Let us outline the proof. We introduce the intermediate vector field $\lambda_0 j^a_\mu j^\mu_a \to A^a_\mu j^\mu_a - \frac{1}{2}\lambda_0 A^a_\mu A^\mu_a$; after integrating over the fermion fields, we find the effective action

$$S = - N_f W\{A\} - \int d^2x \frac{1}{2\lambda_0} \mathrm{tr}(A_\mu A^\mu);$$

$$W\{A\} = i \ln \mathrm{Det}(i\hat{\partial} + \hat{A}); \qquad \hat{A} = A^\mu_a \tau^a \gamma_\mu . \tag{4}$$

It is shown in Ref. 5 that the gauge-invariant functional $W\{A\}$ is a 2D version of the Wess-Zumino action. In a Euclidean space we would have

$$16\pi W\{A\} = -\mathrm{tr}\left\{\int d^2x\, (\mathbf{A}_\mu \mathbf{A}^\mu) + \frac{2}{3} i \int d^3\xi\, \mathbf{A}_\alpha \mathbf{A}_\beta \mathbf{A}_\gamma \epsilon^{\alpha\beta\gamma}\right\}. \tag{5}$$

Here $\mathbf{A}_\alpha(\xi) = a^{-1}\partial_\alpha a (\alpha = 1,2,3)$ is defined in the interior of a 3D sphere Q, whose boundary is the stereographic projection of the x plane under the condition $a = g_+ g_-^{-1}$; we have $A_\pm = A_1 \pm iA_0 = g_\pm^{-1}\partial_\pm g_\pm$ at the boundary. In particular, it follows from (4) and (5) that in the limit $N_f \to \infty$ the fluctuations of A_μ are suppressed, and at $N_f = \infty$ the field A_μ is a purely gauge field: $g_+ = g_- \equiv g, A_\mu = g^{-1}\partial_\mu g$. It follows that at $N_f = \infty$ the actions (1) and (4) are the same. At a finite N_f, we might note, the fermion model has only a single (right) SU(N) invariance. The left symmetry group is reconstructed only at $N_f = \infty$.

3. The fermion model (3) is completely integrable at a finite N_f. Omitting the details of the solution, we give the hierarchy of Bethe equations which arise when periodic boundary conditions for the Bethe wave function are satisfied[1]: The eigenstate of $2\mathcal{N}$ particles belonging to a representation of the symmetry group with the senior weight $[1^{2\mathcal{N}-M_1}, 2^{M_1-M_2}, \ldots, N^{M_{N-1}}]$ is described by the rapidities $\{k_1^{(\pm)} \ldots k_{\mathcal{N}}^{(\pm)}\}\{\lambda_1^{(j)} \ldots \lambda_{M_j}^{(j)}\}$ $(j = 1 \ldots N-1)$, which satisfy the equations

$$\exp(ik_i^{(\sigma)} L) = \prod_{\alpha=1}^{M_1} e_{N_f}(\sigma/\lambda_0 - \lambda_\alpha^{(1)}); \quad \sigma = \pm 1; \quad i = 1 \ldots \mathcal{N},$$

$$\prod_{\sigma=\pm 1} [e_{N_f}(\sigma/\lambda_0 - \lambda_\alpha^{(1)})]^{\mathcal{N}} \prod_{\beta=1}^{M_2} e_1(\lambda_\beta^{(2)} - \lambda_\alpha^{(1)}) = \prod_{\beta=1}^{M_1} (\lambda_\beta^{(1)} - \lambda_\alpha^{(1)}); \tag{6}$$

$$\prod_{\tau=\pm 1} \prod_{\beta=1}^{M_{j+\tau}} e_1(\lambda_\beta^{(j+\tau)} - \lambda_\alpha^{(j)}) = \prod_{\beta=1}^{M_j} e_2(\lambda_\beta^{(j)} - \lambda_\alpha^{(j)}); \; \alpha = 1 \ldots M_j; \; j = 1 \ldots N-1.$$

where $e_n(x) = (x - in/2)/(x + in/2)$. The energy of the state is $E = \Sigma(k_i^{(+)} - k_i^{(-)})$

4. In the thermodynamic limit the solutions of Eqs. (6) group into complexes of order $n = 1 \ldots \infty: \lambda_{\alpha;k}^{(j)} = \lambda_\alpha^{(j)} + i(\frac{n-1}{2} - k)$ $(k = 1, \ldots, n)$. An arbitrary state is determined by the distributions of particles and holes in the bands of the various complexes. We denote by $\rho^j(\lambda)$ and $\rho^j(\lambda)$ the distributions of particles and holes in the band of the N_f complexes corresponding to the jth column of the Young tableau; $l^j_n(\lambda), \tilde{l}^j_n\cdot(\lambda)$ and $r^j_m(\lambda), \tilde{r}^j_m(\lambda)$ are the same for the $n = 1...N_f - 1$ and $m = N_f + 1...\infty$ complexes. After the necessary calculations, we find spectral equations relating the particle and hole distributions. Here are the equations for $N_f = \infty$:

$$\rho^j(\lambda) + R_{jk} * \tilde{\rho}^k(\lambda) + a_m * (r^j_m + l^j_m)(\lambda) = m_j \mathrm{ch}\left(\frac{2\pi\lambda}{N}\right);$$

$$l^j_n(\lambda) + A^{nm} * C^{(N)}_{jk} * l^k_m(\lambda) = a_n * \tilde{\rho}^j(\lambda);$$

$$r^j_n(\lambda) + A^{nm} * C^{(N)}_{jk} * r^k_m(\lambda) = a_n * \tilde{\rho}^j(\lambda); \tag{7}$$

the energy of the state and the numbers $q_j = M_{j+1} + M_{j-1} - 2M_j$, which characterize the symmetry of the state, are

$$\mathcal{E} \equiv (L/N^2)\cdot E = \mathrm{const} + \int d\lambda \sum_j m_j \mathrm{ch}\, {}^{2\pi}/_N \lambda\, \tilde{\rho}^j(\lambda);\ q_j = \int \tilde{\rho}^j(\lambda) d\lambda, \tag{8}$$

and the m_k are given by (2). The asterisk (*) means convolution, and the Fourier transforms of the integral operators used here are

$$R_{jk}(\omega) = \mathrm{th}\frac{|\omega|}{2} A^{(N)}_{jk};\ A^{(N)}_{jk} = 2\mathrm{cth}\frac{\omega}{2} \frac{\mathrm{sh}[(N - \max(jk))\omega/2]\,\mathrm{sh}[\min(j,k)\omega/2]}{\mathrm{sh}\frac{N\omega}{2}},$$

$$A^{nm} = A^{(\infty)}_{nm};\ n, m = 1....\infty;$$

$$C^{(N)}_{jk} = (A^N)^{-1}_{jk} = \delta_{jk} - \frac{1}{2\mathrm{ch}\frac{\omega}{2}}(\delta_{j,k+1} + \delta_{j,k-1});\quad j, k = 1...N-1. \tag{9}$$

$$a_n(\omega) = e^{-|\omega|n/2}.$$

5. The spectral equations embody all the information about the spectrum of particles, the scattering amplitudes, and the thermodynamic properties. In this case we draw the following conclusions from these equations.

1) The ground state of the system is formed by complexes of order N_f.

2) the $\tilde{\rho}^j$, l^j_n, and r^j_n distributions describe excited states: massive physical particles with a mass spectrum (2), which are isotopic multiplets of the $SU(N)\otimes SU(N)$ group. The multiplets transform as antisymmetric $SU(N)\otimes SU(N)$ tensors of rank $j = 1,...,N-1$. Since the jth representation is equivalent to the $(N-j)$th conjugate representation, the jth and $(N-j)$th particles are coupled by a crossing transformation.

3) The jth particle is a bound state of j fundamental particles. The fundamental particles belong to a vector representation. In particular, a fundamental antiparticle may be regarded as a bound state of $N-1$ particles.

4) The two-particle factorized S matrix of the fundamental particles is

$$\mathbf{S}_{11}(\theta) = \exp(i\Phi(\theta))[\sigma_L(\theta) \otimes \sigma_R(\theta)], \tag{10}$$

where $\theta = \ln \frac{s-2m^2+\sqrt{s^2-2sm^2}}{2m^2}$ is the difference between the rapidities of the scattering particles,

$$\sigma_{L(R)} = P^+_{L(R)} + P^-_{L(R)} \frac{\theta + \frac{2\pi i}{N}}{\theta - \frac{2\pi i}{N}} \; ; \quad \Phi(\theta) = \int \frac{d\omega}{\omega} e^{-i\frac{\theta N}{2\pi}\omega} (R_{11}(\omega) - 1),$$

and the operators $P^{\pm}_{L(R)}$ project onto the symmetric (antisymmetric) left (right) scattering channel. The scattering amplitudes of the other particles are found by "blending"[7] the fundamental particles.

We find an unexpected relationship between the chiral field and the Gross-Nevier model[8]:

$$\mathbf{S}_{11}(\theta) = [S^{GN}(\theta) \otimes S^{GN}(\theta)] \frac{\mathrm{sh}\frac{1}{2}\left(\theta - \frac{2\pi i}{N}\right)}{\mathrm{sh}\frac{1}{2}\left(\theta + \frac{2\pi i}{N}\right)}, \tag{11}$$

where S^{GN} is the S matrix of the Gross-Nevier model.[8]

The S matrix[10] is a minimal, unitary, analytic, relativistic, factorized, $\mathrm{SU}(N) \otimes \mathrm{SU}(N)$-invariant S matrix. It was derived in Ref. 9 by a factorized bootstrap method.[10] The scattering amplitudes for the bound states are given in the same paper.

6. The spectral equations can be used to analyze the energy of the ground state of the chiral field as a function of external fields. Let us add to action (1) a term $h_j L^j_0$, or let us add $h_j \bar{\psi}\gamma_0 \hat{H}^j \psi$ to the fermion Lagrangian (3), where $L^j_0 = tr(g^{-1}\partial_0 g \hat{H}^j)$ is the Neter current, and the $H^j = \mathrm{diag}(0,...1 - 1,...,0)$ form the basis of a Cartan subalgebra. We then have $\mathscr{E}(h_j) = min\{\mathscr{E}(q_j) - \Sigma_j h_j \int \tilde{\rho}^j(\lambda) d\lambda\}$. We introduce the functions $\epsilon^{(\pm)}_j(\lambda)$, which satisfy the equations

$$\epsilon^{(-)}_j(\lambda) + R_{jk} * \epsilon^{(+)}_k(\lambda) = h_j - m_j \mathrm{ch}\left(\frac{2\pi}{N}\lambda\right), \quad j = 1 \ldots N-1. \tag{12}$$

It can be shown that we have $\epsilon^{(+)}_j \geqslant 0$ and $\epsilon^{(-)}_j = 0$ if $|\lambda| \leqslant \lambda^{(F)}_j$ or $\epsilon^{(-)}_j \leqslant 0$ and $\epsilon^{(+)}_j = 0$ if $|\lambda| \geqslant \lambda^{(F)}_j$:

$$\mathscr{E}(h_j) = \mathscr{E}(0) + \sum_j \int m_j \epsilon^{(+)}_j(\lambda) \mathrm{ch}\frac{2\pi\lambda}{N} d\lambda. \tag{13}$$

Equation (13) and the conditions $\epsilon^{(\pm)}_j(\pm\lambda^{(F)}_j) = 0$ unambiguously determine $\lambda^{(F)}_j$ and $\mathscr{E}(h_j)$. They are easily analyzed in limiting cases. We assume $h_j = h\delta_{jk}$ and $J_k \equiv \langle L^k_0 \rangle = -\partial\mathscr{E}(h)/\partial h$. The condition $0 < h - m_k \ll m_k$ corresponds to the threshold for the production of a massive particle. At $h \gg m_k$ the expectation value of the current can be calculated by summing the main logarithms of the perturbation theory.

In the two-loop approximation we have $J \propto z^{-1} + O(z)$, where $z^{-1} + \frac{1}{2} N \ln Nz = N \ln \frac{h}{m}$. The same result can be derived from Eqs. (12) and (13).

7. The limit $N = \infty$, in which the chiral field is described by the sum of planar diagrams, is of particular interest. In this limit the S matrix takes the surprisingly simple form

$$S_{11}^{(A_1 A_2 \to A_1' A_2')}(\theta) = (1 + O(N^{-2}))\left\{ I_R \otimes I_L + (I_L \otimes P_R + P_L \otimes I_R)\left(-\frac{2\pi i}{N\theta}\right) + P_R \otimes P_L\left(\frac{2\pi i}{N\theta}\right)^2 \right\},$$

where $A_i = (l_i, r_i)$ $P_R = \delta_{r_1 r_2'} \delta_{r_2 r_1'}$, and $I_R = \delta_{r_1 r_1'} \delta_{r_2 r_2'}$. Curiously, Eqs. (12) can be solved analytically, and $\mathscr{E}(h)$ can be calculated in terms of Bessel functions, so that it becomes possible to study the crossover between the regions of high and low energies. The corresponding analysis will be published separately, as will a solution of the main chiral field for other classical groups.

I wish to thank A. M. Polyakov, A. M. Tsvelik, A. B. Zamolodchikov, N. Yu. Reshetikhin, and A. A. Migdal for useful discussions in various stages of this study.

[1)]In order to solve the fermion model it is necessary to carry out an accurate regularization at high energies. The most systematic approach, involving solution of the nonrelativistic version of model (3), is described in detail in Ref. 6.

[1]A. M. Polyakov, Phys. Lett. **B59**, 87 (1975).
[2]A. M. Polyakov, Phys. Lett. **B72**, 224 (1977).
[3]Y. Y. Goldschmidt and E. Witten, Phys. Lett. **B91**, 392 (1980).
[4]A. M. Polyakov, Acta Unniver. Wratislaviesis No. 436, 53 (1978).
[5]A. M. Polyakov and P. B. Wiegmann, Phys. Lett. B (in press).
[6]A. M. Tsvelik, Preprint No. 7 (1983), Landau Institute of Theoretical Physics; Nucl. Phys. (to be published).
[7]B. Berg, M. Karowski, W. Theis, and H. J. Thun, Phys. Rev. **D17**, 1172 (1978).
[8]B. Berg and P. Weisz, Nucl. Phys. **B146**, 205 (1978); E. Abdalla, B. Berg, and P. Weisz, Nucl. Phys. **B157**, 387 (1979).
[9]P. B. Wiegmann, Phys. Lett. B (to be published).
[10]A. B. Zamolodchikov and Al. B. Zamolodchikov, Ann. Phys. (Paris) **120**, 253 (1979).

Translated by Dave Parsons
Edited by S. J. Amoretty