Rarefaction shock wave in a system with a reversed magnetic field

M. B. Isichenko and K. V. Chukbar
I. V. Kurchatov Institute of Atomic Energy, Moscow

(Submitted 24 November 1983)
Pis'ma Zh. Eksp. Teor. Fiz. **39**, No. 5, 191–192 (10 March 1984)

The relaxation to an equilibrium configuration during the longitudinal contraction of an ideally conducting plasma in a reversed-field system is analyzed. Since the problem is multidimensional, the resulting shock wave may be a rarefaction wave.

One of the questions which arises in research on systems with a reversed magnetic field (see the review by Finn and Sudan,[1] for example) is the relaxation from the initial plasma configuration to the equilibrium configuration. In this letter we analyze this relaxation with the help of a simple two-dimensional model.

We consider a system in which a long, ribbon-shaped, ideally conducting plasma lies halfway between two ideally conducting walls (Fig. 1). The transverse equilibrium (along the y axis) is established more rapidly than the longitudinal equilibrium in this configuration; in other words, the equality of the plasma pressure and the magnetic pressure, $p = H^2/8\pi$, which holds over the greater part of the plasma surface, is violated near its ends. If the thickness of the plasma satisfies $2h \ll 2d$, where $2d$ is the distance between the walls, then we have $H^2/8\pi \gg p$ at the ends. As the magnetic lines of force become constricted, they compress the plasma in the longitudinal direction; this compression is accompanied by a transverse expansion of the plasma to its equilibrium thickness ($h_2 = d/2$; see below), as shown in Fig. 2. This thickening moves along the plasma at a velocity higher than the propagation velocity of small perturbations and is thus of the nature of a shock wave, analogous to a bore at the surface of the liquid.[2]

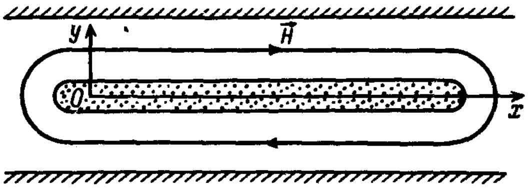

FIG. 1.

The state of the plasma behind the shock front can be found from the conservation laws in the usual way. Transforming to a coordinate system moving with the wavefront, and integrating the *MHD* equations expressing the conservation of mass, momentum, and energy (see Ref. 3, for example) over the volume *abcd* in Fig. 2, we find the system of equations

$$\rho_1 v_1 h_1 = \rho_2 v_2 h_2 \quad ,$$
$$(\rho_1 v_1^2 + p_1)h_1 - \frac{H_1^2}{8\pi}(d-h_1) = (\rho_2 v_2^2 + p_2)h_2 - \frac{H_2^2}{8\pi}(d-h_2), \tag{1}$$
$$\left(\frac{\rho_1 v_1^2}{2} + \frac{\gamma p_1}{\gamma-1}\right)v_1 h_1 = \left(\frac{\rho_2 v_2^2}{2} + \frac{\gamma p_2}{\gamma-1}\right)v_2 h_2$$

(v is the plasma flow velocity, ρ is the density, γ is the adiabatic index, and the subscripts 1 and 2 specify the states respectively ahead of and behind the front). Using the equality $p = H^2/8\pi$, we find from these equations the equation of the shock adiabat:

$$\frac{\rho_2 h_2}{\rho_1 h_1} = \frac{(2h_1 - d)p_1 + (2h_2/(\gamma-1) + d)p_2}{(2h_1/(\gamma-1) + d)p_1 + (2h_2 - d)p_2}. \tag{2}$$

Since the magnetic flux between the ideally conducting plasma and the walls, $\Phi = H(d-h)$, is conserved, we have $p_2 = p_1(d-h_1)^2/(d-h_2)^2$. The plasma flow velocities found from (1) satisfy the necessary conditions $v_1 > c_1$ and $v_2 < c_2$, where c is the propagation velocity of small perturbations: $v_1, v_2 \to c$ as $h_2 - h_1 \to 0$.

Formally, we could substitute any value of h_2 into Eq. (2), but since the plasma behind the shock front must be at longitudinal equilibrium as well as at transverse

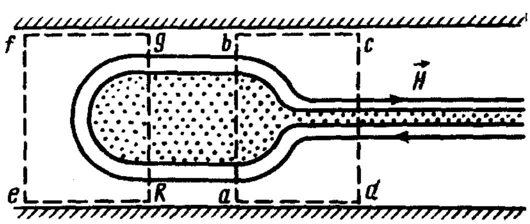

FIG. 2.

equilibrium we find a completely definite value for h_2. To show this, we integrate the momentum conservation equation in the coordinate system with $v_2 = 0$ over the volume *efgh* in Fig. 2. Using the fact that we have $H \to 0$ as boundary *ef* is moved to the left, we find

$$p_2 h_2 - \frac{H_2^2}{8\pi}(d - h_2) = 0,$$

from which we find $h_2 = d/2$. The same relation follows from the circumstance that the energy in the system is taken from the magnetic field.

For an intense shock wave ($h_2 \gg h_1$) we have $p_2 = 4p_1$, and we can rewrite Eq. (2) as

$$\frac{\rho_2}{\rho_1} = \frac{2h_1}{d}\,\frac{3\gamma + 1}{\gamma - 1}. \tag{3}$$

If the ratio $2h_1/d$ is small, the propagation of the shock wave is accompanied by a rarefaction of the plasma (the density per unit length, ρh, and the temperature nevertheless increase).

The front structure of this shock wave can be found analytically only in the approximation of a low intensity. In the linear approximation, in the absence of dissipation, waves of two types can propagate at the plasma surface, with the dispersion relation

$$\omega^2 = \frac{2p}{\rho}\,\kappa k \,\mathrm{cth}\,\kappa h\,[\mathrm{th}\,k(d-h)]^{\nu}, \quad \kappa^2 = k^2 - \omega^2 \frac{\rho}{\gamma p}, \tag{4}$$

where $\nu = -1$ for the antisymmetric mode (with respect to the x axis), which does not alter the plasma volume and $\nu = +1$ for the symmetric mode, which corresponds to the wave under consideration here. The propagation velocity for long waves [kh, $k(d-h) \ll 1$] of the symmetric mode,

$$c^2 = \gamma\frac{p}{\rho}\,\frac{2h}{(2-\gamma)h + \gamma d},$$

becomes equal to the shock-wave velocity (1) in the limit $h_2 - h_1 \to 0$. For nonlinear dispersive waves of small amplitude, viscosity leads to the familiar oscillatory structure of the front (if the viscosity is sufficiently low.[2] In the more important case of a strong shock wave, the front structure cannot be described analytically; we can only assert that if a steady-state structure exists then it is determined not only by the mean free path of the particles in the plasma but also by the characteristic transverse dimension of the system, d.

In summary, it has been shown that in a system with a reversed magnetic field the magnetohydrodynamics of a plasma with an ideal skin effect allows the propagation of some distinctive rarefaction shock waves. Through these waves, the initial plasma configuration relaxes to the equilibrium configuration.

We wish to thank L. I. Rudakov for suggesting this problem.

[1]J. M. Finn and R. N. Sudan, Nucl. Fusion **22**, 1443 (1982).
[2]G. B. Whitham, Linear and Nonlinear Waves, Wiley-Interscience, New York, 1974 (Russ. transl. Mir, Moscow, 1977).
[3]L. D. Landau and E. M. Lifshitz, Élektrodinamika sploshnykh sred (Electrodynamics of Continuous Media), Nauka, Moscow, 1982, §70.

Translated by Dave Parsons
Edited by S. J. Amoretty