Renormalization of topological charge

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The contribution of instantons to the renormalization of the coefficient of the topological charge density in the effective Lagrangian is calculated. The relationship with the quantum Hall effect is discussed.

The effective Lagrangians in certain physical theories may contain a topological charge density. In such theories there are quantum fluctuations of the instanton type, which "revive" the topological charge and which may furthermore lead to its renormalization. In this letter we take up this question in the approximation of an instanton gas, dealing exclusively with the qualitative side of the matter. This phenomenon is extremely general in nature, but its phenomenological importance is not completely clear. For this discussion we have adopted quantum gluodynamics, in which the relevant equations are well known. At the end of this letter we will discuss a possible application in the theory of the quantum Hall effect. We write all of the equations in a Euclidean space; the coefficient of the topological charge is imaginary in this case.

1. The effective Lagrangian $\mathscr{L}(\mu)$, normalized at the point μ, is obtained by definition by averaging the seed Lagrangian over quantum fluctuations with virtualities greater than μ, i.e., over perturbative fluctuations with $q^2 > \mu^2$ and over nonperturbative fluctuations (instanton, etc., fluctuations) with dimensions $\rho < 1/\mu$. We will not take up at this point the physical meaning of $\mathscr{L}(\mu)$; we will concern ourselves with determining its μ dependence. The effective Lagrangian of gluodynamics is

$$\mathcal{L}(\mu) = \frac{1}{4g^2(\mu)} G^a_{\mu\nu} G^a_{\mu\nu} + \frac{i\theta(\mu)}{32\pi^2} G^a_{\mu\nu} \tilde{G}^a_{\mu\nu} . \tag{1}$$

To obtain $\mathscr{L}(\mu' < \mu)$ we need to separate the fields with virtualities greater than and less than μ' in (1); in other words, we need to write G as the sum of $G_{\rm qu}$ (the virtualities between μ' and μ) and $G_{\rm cl}$ (virtualities less than μ') and then integrate over $G_{\rm qu}$. An integration over the perturbative fluctuations alone leads to

$$\begin{gathered} 1/g^2(\mu') = 1/g^2(\mu) + \frac{b}{8\pi^2} \ln \frac{\mu'}{\mu} + \ldots , \quad b = \frac{11}{3} N \\ \theta(\mu') = \theta(\mu) . \end{gathered} \tag{2}$$

Instantons are easily taken into account by making use of an expression for the instanton density in an external field[1] $G^a_{\mu\nu}$:

$$\begin{gathered} d^G(\rho) = \rho^{-5} d(\rho) \exp\left[- \frac{2\pi^2}{g(\rho)} \rho^2 \bar{\eta}^E_{a\mu\nu} G^a_{\mu\nu}(\rho) \right], \\ d(\rho) = C(\rho) \exp(-8\pi^2/g^2(\rho)) . \end{gathered} \tag{3}$$

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The corresponding contribution to the renormalization of $\mathscr{L}(\mu)$ is found by expanding (3) in $\rho^2 G_{\rm cl}$ and retaining terms of up to second order inclusively,[1)] averaging over the instanton orientations, and integrating over $d\rho$ from $1/\mu$ to $1/\mu'$:

$$\frac{1}{\delta(N)} e^{-i\theta} \int_{1/\mu}^{1/\mu'} \frac{d\rho}{\rho} d(\rho) \left(\frac{2\pi^2}{g}\right)^2 (G^2_{\rm cl} - G_{\rm cl}\widetilde{G}_{\rm cl}) \equiv \frac{1}{2} \ln\frac{\mu}{\mu'} D(\mu) e^{-i\theta} (G^2 - G\widetilde{G}). \tag{4}$$

For anti-instantons we would find $e^{+i\theta}(G^2 + G\widetilde{G})$, instead of $e^{-i\theta}(G^2 - G\widetilde{G})$; the appearance of these particular combinations is a consequence of the self-duality of instantons. Using (4), we can put the equations of the "renormalization group" in the following form instead of (2):

$$1/g^2(\mu') = 1/g^2(\mu) + \frac{b}{2\pi^2} \ln\frac{\mu'}{\mu} - 4\ln\frac{\mu}{\mu'} D(\mu)\cos\theta(\mu), \tag{5}$$

$$\theta(\mu') = \theta(\mu) - 32\pi^2 \ln\frac{\mu}{\mu'} D(\mu)\sin\theta(\mu); \qquad D(\mu) > 0.$$

The first of these equations was originally derived by Callan *et al.*[2] We wish to emphasize that $D(\mu)$ is a power function of μ, so that Eqs. (5) cannot be regarded as the actual renormalization-group equations; the specific quantitative dependence $\theta(\mu)$ is not universal (it is different for different processes). The qualitative effect, however, is completely definite:

$$\partial\theta/\partial\mu \sim + \sin\theta, \tag{6}$$

i.e., with decreasing μ, the function $\theta(\mu)$ tends towards an integer multiple of 2π.

2. At this point we make an important digression. There exists an interpretation of θ in terms of a superselection rule.[2] We wish to emphasize that a different θ appears in the effective Lagrangian. The superselection rule asserts that there exists a vacuum $\theta_0: |\Omega\rangle = \Sigma_n e^{in\theta_0}|n\rangle$. The topological charge of the state is $|n\rangle = n$. Here θ_0 is a world constant. The effective Lagrangian is related to the matrix elements between $\Omega_{in} = \Omega(t = -\infty)$ and $\Omega_{\rm out} = \Omega(t = \infty)$. In order to revive θ_0 we consider the amplitude $\langle\Omega_{\rm out}|\Omega_{\rm in}\rangle_Q$ in an external field with a topological charge Q. $\langle Q_{\rm out}|Q_{\rm in}\rangle_Q = \Sigma_{m,n} e^{i\theta_0(n-m)}\langle m|n\rangle_Q \sim \Sigma_k e^{-i\theta_0 k}\langle k|0\rangle_Q$. If instanton transitions are ignored, we are left with a single term from this sum: $e^{-i\theta_0 Q}\langle Q|0\rangle$. If we instead take these transitions into account, we find additional terms:

$$e^{-i\theta_0 Q} < Q'|0> \left[1 + \frac{1}{2}\ln\frac{\mu}{\mu'} D' e^{-iQ_0}(-Q) + \frac{1}{2}\ln\frac{\mu}{\mu'} D' e^{iQ_0}\cdot Q + \ldots\right]$$

$$\to e^{-i(\theta_0 - \ln\frac{\mu}{\mu'} D'\sin\theta_0)Q}$$

From the standpoint of the effective Lagrangian, a matrix element in an external field of this sort contains a factor $e^{-i\theta Q}$, and we see that $\theta = \theta_0 - \ln(\mu/\mu')D'\sin\theta_0$. We might say that the θ_0 from the superselection rule plays the role of a bare θ, while the θ from the effective Lagrangian plays the role of a dressed θ. In our opinion, therefore, the conclusion reached here regarding the renormalizability is consistent with the Hamiltonian approach.

3. The physical interpretation of the $\theta(\mu)$ from the effective Lagrangian seems to us an open question. We recall that scattering processes with momentum transfers q are usually determined by an effective Lagrangian $\mathscr{L}(q)$, while smaller virtualities are unimportant. This assertion, however, comes under doubt in the case of the contribution of many-instanton configurations. Another complicating circumstance is that the topological charge can be revived in the Lagrangian only in processes determined by non-perturbative effects. It is thus not clear to us whether this effect raises the hope that we will learn why the *CP* violation at low energies is slight. At any rate, we should emphasize that this effect is totally unrelated to the instantons as classical solutions; the only point of importance is the assumption of fluctuations with a nonzero topological charge.

4. A field theory with a finite temperature or finite volume is a "pure" situation from the standpoint of the approach based on the introduction of an effective Lagrangian. In such a theory the instanton fluctuations are cut off at the dimensions of the box or reciprocal temperature. A similar situation arises in solid state physics.

In one version of the theory of the quantum Hall effect,[3] the normal conductivity σ_{xx} and the Hall conductivity σ_{xy} of a two-dimensional (2D) film with localization are determined in units of $e^2/2\pi\hbar$ as parameters in an effective Lagrangian of some asymptotically free 2D σ model:

$$\mathcal{L} = \sigma_{xx} \operatorname{Tr} (\partial_\mu \tilde{Q})^2 + i\sigma_{xy} P(\tilde{Q}). \tag{7}$$

Here $P(Q)$ is the topological charge density; the explicit expression for this density is not important for our purposes. Values of σ_{xy} differing by an integer are equivalent. We specify seed conductivities σ_{xx} and σ_{xy} at atomic distances (σ_{xy} varies in proportion to the reciprocal of the magnetic field), while the observable conductivities are parameters of Lagrangian (7), normalized at the point $\mu \sim 1/L$, where L is the dimension of the sample. Experimentally, σ_{xy} tends toward integer values as L is increased. In an effort to explain this fact, Khmel'nitskiĭ[4] suggested that the equations of the renormgroup for σ_{xx} and σ_{xy} have unstable singularities. Figure 1 shows a corresponding phase diagram of the renormgroup equations in the σ_{xx}, σ_{xy} plane. (A special analysis was carried out in Ref. 4 for the infrared region.) The arrows indicate the direction of increasing L. In our approach, this hypothesis is confirmed in a natural

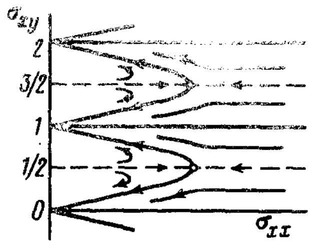

FIG. 1.

way, since the equations for σ_{xx} and σ_{xy} analogous to (5) are

$$\partial\sigma_{xx}/\partial L = \beta(\sigma_{xx}) - \gamma D(\sigma_{xx})\cos(2\pi\sigma_{xy}), \tag{8}$$

$$\partial\sigma_{xy}/\partial L = -\gamma' D(\sigma_{xx})\sin(2\pi\sigma_{xy}),$$

where $D(\sigma_{xx})$ and $\beta(\sigma_{xx})$ are respectively the instanton density and the β function in this model, and γ and γ' are positive constants. In some recent experiments[5] it was found that as the temperature is lowered and the magnetic field raised the quantized value of σ_{xy} may also be fractional: 1/3, 1/5... This effect may be related to the interaction and statistics of the electrons. If so, it does not necessarily have to occur in simplified model (7). However, we do not rule out the possibility that this phenomenon can also be explained by this model, if the Coulomb interaction of electrons is taken into account in some fashion.

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[1)]For $\rho^2 G_{\rm cl}$ to be small it is sufficient that ρ be much smaller than the characteristic value of the field $G_{\rm cl}$, We may then consider fields $G_{\rm cl}$ with a nonzero topological charge, and we are justified in retaining the topological charge in (4).

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