Method of eigenfunctions and noneigenfunctions in the theory of the beam-plasma instability of a spatially bounded electron beam

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It is shown by numerical calculations that it is possible to transform an arbitrary initial perturbation in a plasma with a bounded electron beam into an eigenfunction. The physical reasons for the establishment of this solution and the conditions for the applicability of the method of eigenfunctions and noneigenfunctions in the theory of the beam-plasma instability of bounded electron beams are studied.

An important problem in the physics of the beam-plasma instability is how the transverse geometric dimensions of the beam affect the instability growth rate. An effective method for studying this problem, the method of noneigenfunctions, was proposed in Ref. 1. This method has the disadvantage that in order to achieve the completeness and orthogonality of the wave functions used there it is necessary to satisfy the condition $\Delta_0 q \gg 1$, where Δ_0 is the width of the wave packet, and q is the characteristic transverse wave number. This disadvantage becomes particularly obvious when we note that the maximum growth rate of the beam-plasma instability is reached for perturbations with $q = 0$.

In the present letter we attempt to circumvent this difficulty, and we analyze the evolution of initial perturbations with $q = 0$.

According to Ref. 1, the equation which is the starting point and which describes the plasma waves of a plasma of density n_0 under the influence of a spatially bounded electron beam without a magnetic field is

$$\left(\frac{\partial}{\partial t} - i\,\frac{C_s^2}{2\omega_0}\Delta_\perp\right)\left(\Delta_\perp - k_z^2\right)\Psi = -\,2\pi\omega_0\,\frac{e^{*\,2}}{m}\,\frac{2}{V_T^2}\left(i + \sqrt{\frac{\pi}{2e}}\right) n_b \Psi, \tag{1}$$

where ω_0 is the plasma frequency, e^* is the electron charge, m is the electron mass C_s is the speed of sound in the plasma, V_T is the thermal velocity spread of the beam electrons, and n_b is the density of the beam electrons, which we write in the form $n_b = n_{b_0}\exp\{-(r^2/2\Delta_b^2)\}$ (which corresponds to a cylindrical beam with a Gaussian density profile). In deriving (1) we assumed that the beam is directed along the z axis, and we accordingly adopted a potential perturbation

$$\Psi = \Psi(t, r)\exp\left\{-i\omega_0 t - i\,\frac{C_s^2}{2\omega_0}k_z^2\,t + ik_z z\right\}.$$

In all the calculations below we assume $k_z = \omega_0/V$, where V is the directed velocity of the beam electrons. We will solve Eq. (1) by the initial-perturbation method; i.e., we will construct a Cauchy problem for Eq. (1):

$$\Psi(t = 0,\ r) = \Psi_0(r). \tag{2}$$

By virtue of the cylindrical symmetry of the beam, the initial perturbations should also be chosen to be cylindrically symmetric. To solve (1) with initial condition (2) we expand $\Psi(t,r)$ in a series in Laguerre functions:

$$\Psi(t, r) = \sum_{n=0}^{\infty} a_n(t)\Psi_n = \sum_{n=0}^{\infty} a_n(t)L_n(r^2/\Delta_0^2)\exp\left\{-\left(r^2/2\Delta_0^2\right)\right\}, \tag{3}$$

where Δ_0 is an arbitrary parameter, and the L_n are the Laguerre polynomials. Substituting (3) into Eq. (1) and initial condition (2), we find a system of ordinary differential equations for the $a_n(t)$:

$$\sum_{m=0}^{\infty} A_{n,m}\frac{da_m}{dt} = \sum_{K=0}^{\infty} V_{n,K}\,a_K, \tag{4}$$

with the initial conditions

$$a_n(t = 0) = a_{n0}\ , \tag{5}$$

where $A_{n,m}$ is the tridiagonal matrix

$$A_{n,m} = \int_0^{\infty} \Psi_n\left(\Delta_\perp - k_z^2\right)\Psi_m\,dr^2,$$

and the matrix $V_{n,m}$ is, correspondingly,

$$V_{n,m} = \int \Psi_n\left\{i\,\frac{C_s^2}{2\omega_0}\Delta_\perp(\Delta_\perp - k_z^2) - 2\pi\omega_0\,\frac{e^{*\,2}}{m}\,\frac{2}{V_T^2}\left(i + \sqrt{\frac{\pi}{2e}}\right)n_b(r)\right\}\Psi_m\,dr^2.$$

System (4) with initial conditions (5) has been integrated numerically. In choosing the initial conditions we set all the a_{n_0} except a_{00} equal to zero (a Gaussian packet), and we restricted the number of functions Ψ_n to 10–12 (since the amplitude of the 10th to 12th

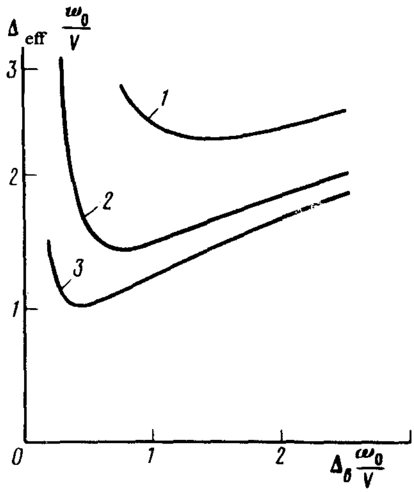

FIG. 1. The width of the eigenfunction, $\Delta_{\rm eff}$, vs the beam radius Δ_b for various parameters $\kappa=(C_s^2V_T^2n_0)/(V^4n_{b_0})$: 1—$\kappa=1.5$; 2—$\kappa=3\times10^{-1}$; 3—$\kappa=6\times10^{-2}$.

function turns out to be four or five orders of magnitude smaller than the amplitude of the 0 th function in the calculations). In studying the solutions (4) over broad ranges of the plasma and beam parameters, we found that the solution of problem (4), (5) converges on an eigenfunction. The physical meaning of this solution is easily seen from the following arguments: Any arbitrary initial perturbation of the Gaussian-packet type with dimensions much smaller than the transverse dimensions of the beam must "spread out" without limit because of dispersion. In the opposite limiting case, i.e., when the beam width is much smaller than the perturbation width, only the part of the perturbation in the beam volume experiences the growth caused by the beam-plasma instability; the rest of the perturbation is essentially unaffected by this instability. As a result, the effective dimensions of the perturbation decrease, despite the dispersion. The competition between these processes gives rise to an equilibrium perturbation, whose shape remains constant over time and which grows as a whole under the influence of the beam. It is natural to treat such a perturbation as an eigenfunction of Eq. (1). Figure 1 shows the effective width of the equilibrium perturbation, $\Delta_{\rm eff}$, as a function of the beam radius Δ_b, where

$$\Delta_{eff}^2=\frac{1}{2\Psi^*(0)\Psi(0)}\int_0^\infty \Psi^*(r)\,\Psi(r)\,dr^2. \tag{6}$$

We see that in the limit $\Delta_b\to0$, we have $\Delta_{\rm eff}\to\infty$. The reason for this result is that the beam "power" falls off with decreasing Δ_b, and the ability of the beam to compress the perturbation accordingly fades. It follows that in the case $\Delta_{\rm eff}\gg\Delta_b$ and with a small growth rate for the eigenfunction we should use the method of initial perturbations in the spirit of Refs. 1 and 2 to analyze the beam-plasma instability. In the case $\Delta_{\rm eff}\lesssim\Delta_b$, on the other hand, it is sufficient to analyze the behavior of the eigenfunction. The width of the eigenfunction should also be followed when we are interested in the effect of plasma inhomogeneities outside the beam on the growth rate for the beam-plasma instability (see Ref. 3, for example).

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