Theory of electron transport in a strong magnetic field

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A simplified system of hydrodynamic equations, which is valid for an arbitrary ratio of the transverse scale of inhomogeneity to the Larmor radius of ions, is proposed for the description of nonlinear low-frequency plasma oscillations in a tokamak and in other systems with a strong magnetic field. General properties of this system are discussed and the anomalous coefficients of electronic thermal conductivity and of plasma diffusion are estimated.

Experiments have shown that a complicated activity of a collective nature is always present in a tokamak plasma, confined and stabilized by a strong magnetic field. This activity manifests itself in low-frequency oscillations with finite amplitude and in the anomalously high diffusion and electronic heat transfer. To describe these phenomena we can use simplified equations, in which the presence of a very strong magnetic field is taken into account explicitly. Within the framework of magnetic hydrodynamics, such equations were obtained by us[1] by expanding the MHD equations in inverse powers of the longitudinal magnetic field B_0. These equations turn out to be very useful for describing an entire series of collective phenomena in a tokamak plasma: reconnection of the magnetic field on the mode $m = 1$,[2] nonlinear tearing modes and tearing instability.[3–8] These equations are, however, inadequate for finding the diffusion and thermal conductivity due to small-scale turbulence, and they must be generalized to scales considerably smaller than the average Larmor radius of ions, ρ_i.

To obtain the equations of nonlinear dynamics in a strong magnetic field, we shall explicitly take into account the fact that the longitudinal component of the magnetic field B_0 is much greater than the transverse component $B_\perp$. We shall model the geometry of the tokamak by a straight cylinder of length $L = 2\pi R$, where R is the large radius of the torus. We orient the z axis of the coordinate system along B_0. The field B_0 remains essentially constant with slow flows of the low-pressure plasma, so that the condition div $\mathbf{B} = 0$ reduces to div $\mathbf{B}_\perp = 0$, which permits introducing the stream function ψ:

$$\mathbf{B}_\perp = [\mathbf{e}_z \nabla\psi], \tag{1}$$

where $\mathbf{e}_z$ is a unit vector along the z axis. In accordance with (1), the electric field can be represented in the form

$$\mathbf{E} = -\nabla\phi + \frac{\mathbf{e}_z}{c}\frac{\partial\psi}{\partial t}\,. \tag{2}$$

To find the equation for ψ, we shall use the drift kinetic equation for electrons

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$$\frac{\partial f}{\partial t} + v_{\|}(\mathbf{b}\nabla)f + \frac{c}{B_0}[\mathbf{e}_z\,\nabla\phi]\,\nabla f + \frac{e}{m}(\mathbf{b}\,\nabla\phi - \frac{1}{c}\frac{\partial\psi}{\partial t})\frac{\partial f}{\partial v_{\|}} = 0, \tag{3}$$

where $\mathbf{b} = \mathbf{B}/B_0$. In a tokamak the distribution function f is close to the Maxwellian function f_0, so that we can set $f = f_0 + \tilde{f}$, where $\tilde{f}$ is small. For this reason Eq. (3) can be linearized relative to it. The equation for ψ can be obtained by multiplying (3) by $v_{\|}$ and integrating with respect to $v_{\|}$.

The equation for $\tilde{f}$ which can be solved by a standard Fourier transformation method, gives

$$\frac{\partial\psi}{\partial t} = c\frac{\mathbf{B}}{B_0}(\nabla\phi - \frac{\nabla p_e}{en}) + \frac{c^2\hat{\eta}}{4\pi}\Delta_{\perp}\psi, \tag{4}$$

where $\hat{\eta}$ is the collisionless resistivity of the plasma, which in the Fourier representation is found from the relation

$$\eta^{-1}_{\mathbf{k}\omega} = \sigma_{\mathbf{k}\omega} = -i\frac{e^2 n\omega}{T_e k_{\|}}\{1 + \frac{\omega}{k_{\|}v_e}Z(\frac{\omega}{k_{\|}v_e})\}. \tag{5}$$

Here $v_e = \sqrt{2T_e/m}$, m is the mass of the electron, and Z is the so-called dispersion function.[8] Since in what follows we shall need only the real part of $\hat{\eta}$, we set approximately

$$\hat{\eta} \cong \mathrm{Re}\,\hat{\eta} \approx \begin{cases} \dfrac{mk_{\|}v_e}{e^2 n} & \text{for} \quad \omega^2 \leqslant k_{\|}^2 v_e^2, \\ 0 & \text{for} \quad \omega^2 > k_{\|}^2 v_e^2, \end{cases} \tag{6}$$

where $k_{\|}$ is the characteristic average wave number for perturbations. Because of the toroidal nature of a tokamak, a satellite addition $\pm 1/qR$ appears in $k_{\|}$.[1)] This addition is the main contribution to Re $\hat{\eta}$ if $qR < v_e/\omega$. We are interested in oscillations which have the form of excitations that are strongly elongated along the magnetic field. In such oscillations the field-aligned motion of ions can be ignored.

The equation of continuity can be obtained by integrating (3) along $v_{\|}$,

$$\frac{\partial n}{\partial t} + \frac{c}{B_0}[\mathbf{e}_z\,\nabla\phi]\nabla n = \frac{c}{4\pi e}(\mathbf{b}\nabla)\Delta_{\perp}\psi. \tag{7}$$

Here we replaced $nv_{\|e}$ by $-j/e$, where the field-aligned component of the current density, according to (1), is

$$j = \frac{c}{4\pi}\Delta_{\perp}\psi. \tag{8}$$

For ions it is sufficient to examine plane-parallel flows in a potential electric field $\mathbf{E} = -\nabla\phi$. Experiments have shown that small-scale fluctuations of the plasma density in tokamaks are nearly isotropic. We may therefore assume tentatively that the ion oscillations occur against a uniform background, $n_0 = \text{const}$. In this case, however, the coupling between the density fluctuations $\tilde{n}$ and ϕ, which follows from the kinetic equation for ions, is well known,

$$\tilde{n} = -\frac{e}{T_i} n_0 \{1 - \exp(\rho_i^2 \Delta_\perp) I_0(-\rho_i^2 \Delta_\perp)\} \phi, \tag{9}$$

where T_i is the temperature of the ions, $\rho_i^2 = T_i/M\omega_{Bi}^2$, $\omega_{Bi} = (eB_0)/(Mc)$, M is the mass of an ion, and I_0 is the Bessel function of an imaginary argument. Substituting $n = n_0 + n$ into (7), we obtain

$$Mn \left\{\frac{\partial \Gamma}{\partial t} + \frac{c}{B_0}[\mathbf{e}_z \nabla\phi] \nabla\Gamma \right\} = \frac{\mathbf{B}\nabla}{4\pi} \Delta_\perp \psi, \tag{10}$$

$$\Gamma = -\rho_i^{-2} \{1 - \exp(\rho_i^2 \Delta_\perp) I_0(-\rho_i^2 \Delta_\perp)\} \frac{c\phi}{B_0}. \tag{11}$$

Equations (4), (7), and (10) are the basic system of equations for dynamics of a plasma in a strong magnetic field. For the temperature T_e, depending on the desired accuracy of the description, we can either set approximately $\mathbf{b}\nabla T_e = 0$ or we can use kinetic equation (3).

To clarify what happens with magnetic surfaces, it is convenient to examine another equation for the surface Φ which moves together with the plasma,

$$\frac{d\Phi}{dt} \equiv \frac{\partial \Phi}{\partial t} + \frac{c}{B_0}[\mathbf{e}_z \nabla\phi] \nabla \Phi = 0. \tag{12}$$

If initially we choose $\overline{\mathbf{B}}\nabla\Phi = 0$, then in the ideal case, $\hat{\eta} = 0$, the lines of force will lie on this surface in the future as well. In other words, with $\hat{\eta} = 0$, Eq. (12), together with (4), leads to the condition $(d/dt)(\mathbf{B}\nabla)\Phi = 0$. If, however, $\hat{\eta} \neq 0$, we would have

$$\frac{d}{dt}(\mathbf{B}\nabla)\Phi = \frac{c^2}{4\pi}[\nabla(\hat{\eta}\Delta_\perp \psi)\nabla\Phi]_z \tag{13}$$

for any surface that satisfies Eq. (12). Now the lines of force lying on the surface $\Phi = \text{const}$ at $t = 0$ begin to "penetrate" through it with time, giving rise to the transport of particles and heat across the plasma column.

Let us consider the equation of continuity (7). It is easy to verify that at $\hat{\eta} = 0$ it conserves the number of particles within the surface, $\Phi = \text{const}$, i.e., there is no transport. It can also be shown that because $B\nabla T_e = 0$, there is also no heat flow. Thus the convective transport of heat and particles in these oscillations can occur only due to the finite conductivity.

Using Eq. (12) we can obtain an equation for the flow of particles and heat. However, even without calculations, it is evident that the only scale for the coefficients of thermal diffusivity[7] and diffusion is the coefficient of "magnetic" diffusion $c^2\hat{\eta}/4\pi$, which characterizes the rate at which the lines of force pass through the plasma, consistent with Eq. (13):

$$D \sim \chi_e \sim \frac{c^2 \hat{\eta}}{4\pi}\epsilon \approx \frac{c^2}{\omega_{Pe}^2}\frac{v_e}{qR}\epsilon, \tag{14}$$

where ϵ appears in this equation due to the ballooning noted previously.

These quantities have this scale in tokamaks with resistive heating and low plasma pressure, if there is no reason to expect very high oscillations.

[1)]Whose amplitude is a factor of ϵ ($\epsilon = r/R$) lower than the amplitude of the fundamental harmonic.

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