Supersymmetry in the problem of an electron in a nonuniform magnetic field

L. É. Gendenshteĭn
Khar'kov Physicotechnical Institute, Academy of Sciences of the Ukrainian SSR

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The Pauli and Dirac equations for an electron are super-symmetric in an arbitrary two-dimensional magnetic field. The vacuum can be degenerate with respect to an *infinite-dimensional representation* of a group which is not the symmetry group of the Hamiltonian. Supersymmetry also occurs in a three-dimensional field with a certain parity. The parity plays the role of the fermion charge in this case.

1. Supersymmetry, a new type of symmetry of physical systems, which includes anticommutation relations as well as commutation relations, was proposed[1] in a study of models in quantum field theory and still awaits experimental confirmation in this region. As it turns out, however, supersymmetry is a physical symmetry in the important and widely used problem of an electron in a nonuniform magnetic field which is directed along one axis (the z axis) and which depends in an *arbitrary* way on the two other coordinates (x and y).

This problem has supersymmetry because the Hamiltonian can be factorized; this possibility was discovered and exploited in Refs. 2 to lower the order of the differential equation determining the ground states. It was shown in the same papers that the factorization of the Hamiltonian may cause the ground state to be degenerate even in a nonuniform magnetic field, despite the loss of the Hamiltonian's translational invariance. In the present letter we find a group under an (infinite-dimensional) representation of which the ground state may be degenerate, and we derive a condition under which this degeneracy occurs.

2. This analysis may be pertinent not only to the particular problem of an electron in a nonuniform magnetic field but also to an analysis of the general structure of supersymmetry theories: Here we have a supersymmetric quantum mechanics with *two* boson degrees of freedom (x, y) and *one* fermion degree of freedom (the spin). This difference between the numbers of boson and fermion degrees of freedom gives rise (without disturbing the existence of the supersymmetry itself!) to some qualitatively new features which are not found in Witten's one-dimensional, (1D) supersymmetric quantum mechanics: Specifically, a supersymmetric vacuum with an energy $E = 0$ can be *degenerate*, despite the fact that the Hamiltonian has no "internal" symmetry of any sort beyond the supersymmetry itself. In particular, the degree of degeneracy of the vacuum can be infinite, and it can furthermore be degenerate with respect to the representation of the symmetry group of the *free* Hamiltonian, if we work by analogy with field theory. It is important to note that this property of the vacuum is stable with respect to arbitrary continuous deformations of the superpotential which do not alter its global characteristics (more on this below); in other words, this property is more of a topological than a purely group property. Another curious point is that this phenomenon occurs in a system with a finite number of degrees of freedom.

3. The Pauli Hamiltonian[1)] for a spin-1/2 particle in a 2D magnetic field is

$$\hat{H} = (P_x^2 + P_y^2 + i\mu\sigma_3[P_x, P_y])/2, \tag{1}$$

where $P_x = p_x - eA_x$, $P_y = p_y - eA_y$, the σ_i are the Pauli matrices, μ is the magnetic moment, expressed in Bohr magnetons, and $\hbar = c = m = 1$.

If the particle does not have an anomalous magnetic moment, i.e., if $\mu = 1$, then Hamiltonian (1) has a supersymmetry with the algebra

$$\{Q_i, Q_k\} = \delta_{ik}\hat{H}; \quad [Q_i, \hat{H}] = 0; \quad i, k = 1, 2 \tag{2}$$

and with the supersymmetry generators $Q_1 = (\sigma_1 P_x + \sigma_2 P_y)/2$, $Q_2 = (\sigma_2 P_x - \sigma_1 P_y)/2$.

Supersymmetry is intimately related to the principle of a minimal incorporation of the interaction with the electromagnetic field as a *gauge* principle, and the requirement that the nonrelativistic Pauli Hamiltonian be supersymmetric unambiguously determines the magnetic moment of the electron, which is equal to the Bohr magneton. This value of the magnetic moment of the electron is known to follow from the principle of a minimal incorporation only in the relativistic Dirac theory.

The existence of two anticommuting integrals of motion—the supercharges Q_1 and Q_2—leads to a twofold (or even-number) degeneracy of all the levels with $E > 0$. This is a generalization of the well-known twofold degeneracy of Landau levels in a uniform field.[2,4] The energy-degenerate states—superpartners—are converted into each other by the supersymmetry generators $Q_\pm = Q_1 \pm iQ_2$, which *simultaneously* change the direction of the spin and the dependence of the wave function on x and y. An important distinction between a nonuniform field and a uniform one is that in a uniform field the superpartner states belong to the same series of orbitals, while in a nonuniform field they belong to two distinct series.

4. By virtue of (1) and (2), the problem of determining the states with an energy $E = 0$, which we will call "vacuum states," reduces to the problem of solving a first-order equation for single-component wave functions[2)]:

$$P_+ \psi = -(2i\partial/\partial\bar{z} + eA(z, \bar{z}))\psi(z, \bar{z}) = 0, \tag{3}$$

where $z = x + iy$, $A = A_x + iA_y$, $P_+ = P_x + iP_y$. It follows from (3) that we have $\psi = f(z)\exp(-\phi)$, where $f(z)$ is an integral function, and ϕ satisfies the equation[2] $\Delta\phi = eB_z(x, y)$. We adopt the gauge $\mathrm{div}\mathbf{A} = 0$.

The degree of degeneracy of the vacuum is equal to the number of linearly independent integral functions $f(z)$ corresponding to the quadratically integrable ψ. If the total magnetic flux is infinite, then the vacuum is infinitely degenerate, despite the fact that the magnetic field may have no regularity of any sort.

5. To determine the group under whose representation the vacuum is degenerate we note that this group must be a symmetry group of the operator P_+ but not necessarily of the Hamiltonian $\hat{H}$. The operator P_+ commutes with the operators $a = \partial/$

$\partial z + i\overline{A}(z, \bar{z})/2$ and z, which obey a commutation relation that determines the Heisenberg-Weyl algebra: $[a, z] = 1$. For the operators a and z to correspond to a representtion of the Heisenberg-Weyl group with ground-state wave functions of the type $f(z)\exp(-\phi)$ it is sufficient that the norm, determined for the integral functions by the scalar product $(u, v)_\phi = \int \overline{u(z)}v(z)\exp(-2\phi)d\bar{z}dz$, be equivalent to the corresponding norm for a uniform field,[3] $(u, v)_0 = \int \overline{u(z)}v(z)\exp(-eB_0\bar{z}z/2)d\bar{z}dz$[3]. The representation is then equivalent to a unitary representation. The irregularity of the field is factorized for the ground-state wave functions in the form of a factor $\exp(-\phi)$ functions, by analogy with the procedure in a uniform field, in which we have $\phi = eB_0\bar{z}z/4$. A distinction between the cases of uniform and nonuniform fields is that in a uniform field wave packets having different localizations (different degeneracies with respect to the center of the orbit) but an identical shape correspond to different ground states, while in a nonuniform field the shape of the wave packet also changes upon a displacement.

6. The Pauli Hamiltonian for a particle with a magnetic moment equal to the Bohr magneton also has supersymmetry in a 3D field if $\mathbf{A}(-\mathbf{r}) = -\mathbf{A}(\mathbf{r})$. In this case the operators which form the supersymmetry algebra (2) are $Q_1 = \vec{\sigma}(\mathbf{p} - e\mathbf{A})/2$ and $Q_2 = iIQ_1$, where I is the parity operator. As the supersymmetry generators $Q_\pm = Q_1 \pm iQ_2 = (1 \pm I)Q_1$ act on states with a definite parity, they either annihilate them or switch their parity. In this example we are thus seeing a generalization of the concept of a "fermion degree of freedom": The roles of the states with "fermion numbers" 0 and 1 are played by states with a definite parity.

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[1]We are dealing with the nonrelativistic case everywhere below, but all the results can be applied, with the obvious modifications, to the relativistic Dirac equation, with the one exception that the ground states corresponding to a massless fermion are genuine null modes.
[2]We adopt $e\Phi > 0$; we write the magnetic flux as $\Phi = \int B_z(x, y)dxdy$; and we consider fields which have no singularities and which conserve direction.
[3]This condition holds for the case $B_z(x, y) = B_0 + b(x, y)$ of physical interest, if the flux of the irregular part, $b(x, y)$, is zero on the average, while the field $B_z(x, y)$ does not vanish.

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