Turbulence of a gravitational field near a cosmological singularity

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Certain considerations suggest an unbounded development of small-scale structure in the course of a cosmological collapse. As a result, our understanding of the conditions under which a singular point is approached may be altered substantially.

A singular point in time remains a fundamental problem in cosmology (and perhaps in theoretical physics in general). As the years go by, the inevitability of this phenomenon and its significance are coming to be appreciated progressively more widely (see Ref. 1 as an example of a recent result in support of this assertion). The

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behavior of a gravitational field (in vacuum) in the asymptotic proximity of this singularity in the general nonuniform case is not completely clear. Some studies, whose results are summarized in Refs. 2, show that the chaotic regime of approaching the singular point is exceedingly complex. Nevertheless, the process can be followed completely in the particular case of a uniform space. However, the arguments presented below force us to believe that the nature of the singularity, in general, is qualitatively more complex than in uniform models, and a fundamentally new approach must be taken in order to solve the problem completely.

The solution proposed as a general solution in Ref. 2 is described by an oscillatory regime of the same nature as in a type-IX uniform model. A basic element of this solution is the concept of a "Kasner epoch." The metric of each such epoch is

$$-ds^2 = -dt^2 + (t^{2p_1} l_\alpha l_\beta + t^{2p_2} m_\alpha m_\beta + t^{2p_3} n_\alpha n_\beta) dx^\alpha dx^\beta, \qquad (1)$$

where the vectors l_α, m_α, n_α and the Kasner indices p_1, p_2, p_3 depend on the three spatial coordinates only. The time evolution of the solution (as $t \to 0$) is made up of an infinite alternation of these epochs with laws governing the interchange of the indices p_1, p_2, p_3 and the directions of the Kasner axes l_α, m_α, n_α which are the same as the laws in a type-IX model. There is a fact of fundamental importance here: A necessary condition for the construction of solutions of this sort is that all three-dimensional functions which are contained in metric (1) have a common and unique length scale over which these functions change substantially. As a result, all the derivatives of the components of the metric with respect to spatial variables in Einstein's equations can be approximated by simple products of these components and a characteristic wave number. In this manner we obtain ordinary equations in time which are of the same form as in a uniform type IX model. This is the reason for the agreement of the pattern of oscillations in a homogeneous model and that in the generalized solution described in Ref. 2.

It follows that both a type-IX model and its generalization contain an oscillatory regime with only a single spatial scale, and the size of this scale is arbitrary. No physical conditions of any sort distinguish it from all other scales. On the other hand, it is known that, in a nonlinear system with an infinite number of degrees of freedom, a regime of this sort is unstable with respect to a partial decay into oscillations of smaller scales. In general, among small perturbations with an arbitrary spectrum there are always some whose amplitudes grow, drawing energy from the main process. As a result, we find a complex picture of multiscale motions with a certain energy distribution and with an exchange of energy among oscillations of various scales. The only case in which this picture does not arise is that in which the physical conditions of the problem rule out a growth of small-scale oscillations. For this purpose there must be a natural physical length which determines the minimum scale at which energy is extracted from the system of dynamic degrees of freedom (as is the case, for example, in a sufficiently viscous liquid). For a system such as a gravitational field in vacuum, however, there is no internal physical scale, so there is nothing to prevent the growth of oscillations of arbitrarily small scale.

Interestingly, this conclusion, which has been drawn from general considerations, also finds support in a direct quantitative analysis of cosmological evolution, again with the help of a representation of an alternation of Kasner epochs and an infinite process of the conversion of Kasner indices. It can be shown that the spatial gradients of the indices increase rapidly and without bound as the singularity is approached. The gradients of all other components of the metric also increase, because they are coupled with the indices through the field equations. This observation was first made by Kirillov and Kochnev,[3] who applied the results to a study of a completely different case. This phenomenon was observed independently five years later by Montani[4] in a graduation thesis at the University of Rome. Montani called this phenomenon "fragmentation" (alternatively, it is a "formation of cells" in the terminology of Kirillov and Kochnev). If we do not trace the correspondence between the Kasner indices and the Kasner axes (this correspondence is unimportant at the moment), and if we focus exclusively on the evolution of three numerical quantities which give a triad of indices in each epoch in some order or other, then this effect can be seen easily by examining the evolution of the parameter u, in terms of which the Kasner indices are expressed:[2]

$$p_1=\frac{-u}{1+u+u^2},\quad p_2=\frac{1+u}{1+u+u^2},\quad p_3=\frac{u(1+u)}{1+u+u^2}. \tag{2}$$

As u varies from 1 to ∞, the numerical values of the indices take on all possible values on the intervals

$$-\frac{1}{3}\leqslant p_1\leqslant 0,\quad 0\leqslant p_2\leqslant\frac{2}{3},\quad \frac{2}{3}\leqslant p_3\leqslant 1. \tag{3}$$

Values of u less than unity contribute nothing new, since the quantities in (2) are invariant (within an interchange of no interest here) under the transformation $u\to 1/u$. Analysis of the equations[2] shows that the evolution of the parameter u is described by the infinite sequence

$$u_1,u_1-1,\dots,x_1\to\frac{1}{x_1}\equiv u_2,u_2-1,\dots,x_2\to\frac{1}{x_2}\equiv u_3,u_3-1,\dots . \tag{4}$$

Written in this form, this equation means that if the value of u in the initial epoch is some number u_1 greater than unity, then the value of the parameter u in subsequent epochs becomes smaller by one in each case. The pattern continues in this manner until the parameter reaches a value x_1 which is less than one. The latter value, however, is equivalent to a value $1/x_1$, by virtue of the invariance which we just mentioned. The evolution from this point on is precisely the same as the earlier evolution, but with the new initial value of the parameter, $u_2=1/x_1$ (and so forth). In uniform models this evolution is the same for the entire space; i.e., the Kasner indices do not depend on the spatial coordinates. In general, the initial value of the parameter u differs from point to point. This circumstance, combined with (4), leads to a singularity in the course of the evolution—i.e., to an unbounded growth of the spatial derivatives of the metric—no matter how close the initial distribution of the parameter u is to a uniform distribution.[3,4] As an example, let us consider the case in which the function $u_1(x,y,z)$ is given. This function is continuous throughout the space, taking on values from 1 to ∞ (we assume that the latter value is reached at spatial infinity). In the initial epoch, the entire three-dimensional space then constitutes a single natural cell, in which the range of the indices spans the intervals in (3) completely and only once (we let this be the definition of a cell). In the next epoch, according to (4), a spatial region arises. In this region the parameter u varies between zero and one, and (after the substitution $u \to 1/u$) the parameter varies between 1 and ∞. Outside this region, throughout the rest of the space, the parameter u varies between 1 and ∞, as before. We thus now have two cells, instead of the one cell in the preceding epoch. In the course of the transition to the third epoch, the same mechanism leads to a splitting of each of the two cells into two new ones, etc. As the result of an infinite evolution of parameter (4), the number of cells goes off to infinity, and the size of the cells approaches zero. Since the indices within each cell must take on all possible values, their spatial gradients must tend toward infinity. This process demonstrates (in a different formulation) the tendency of a gravitational field toward an unbounded self-excitation of progressively smaller-scale oscillations as the cosmological singularity is approached.

The evolution of a single excitation with an arbitrarily large wave number does not by itself pose any problems for the approach taken in Ref. 2. The description of the key element of the oscillatory regime—the alternation of Kasner epochs—remains valid for any length scale. The problem is that this method is incapable of dealing with the consequences of the interaction of a large number of excitations, with a broad spectrum of possible wave numbers. By virtue of the nonlinearity of the theory, this interaction is governing, and the process cannot be thought of as a superposition of oscillations with different length scales, each described by the mechanism studied in Ref. 2.

We thus see that the possible existence of self-organization phenomena and so-called restoration, in the course of which an interaction between modes restores the system to a simpler motion (approximately a single-scale motion, for example), acquires a greater importance. Furthermore, we need to understand that the idea of an unbounded (on the side of large wave numbers) self-excitation of small-scale oscillations nevertheless follows from general considerations, since the support found for this idea on the basis of the Kirillov–Kochnev–Montani observation is of only limited meaning (this observation is itself based on the concept of an alternation of Kasner epochs, and this concept loses its validity after the appearance of a sufficiently developed many-scale motion). To clarify these questions, we must consider the nature of the migration of energy through the spectrum. Fortunately, the situation in this regard is simplified by the circumstance that a collapsing gravitational field has a specific effect: a continuous pumping of energy into a system of oscillatory degrees of freedom. The pumping rate itself increases without bound as the singularity is approached. To see that this is the case, we need to single out the determinant of the metric as a common multiplier in the metric-tensor matrix in Einstein's equations in a synchronous system, while the other components (a matrix with a unit determinant) should be assigned two oscillatory degrees of freedom. We then find that the equations describe the evolution of oscillatory components under the influence of a continuously increasing flux of energy from the exterior as the singularity is approached (this increase is a consequence of the Landau–Raichoudhury theorem regarding a monotonic tendency of the determinant toward zero). From the physical standpoint, this effect is of the same nature as the conversion of the potential energy of a system of collapsing particles into the kinetic energy of their relative motion. Since we know that there is nothing other than oscillations in this case, and there is no special length scale, this influx of energy solves the problem of the energy flow through the spectrum in a first approximation: The continuous pumping of energy toward infinitely small scales should be dominant in this energy flow. Under these conditions, a restoration is improbable, and it may be a continuous excitation of progressively new small-scale modes which determines the behavior of the system. As a result, the asymptotic state of the field near the singularity might be called an "infinitely developing gravitational turbulence." Such a state cannot be described on the basis of the classical concept of a general solution of differential equations. Instead we must look at the determination of exclusively statistical characteristics in terms of random-field theory and some type of phenomenological description.

[1] A. Vilenkin, Phys. Rev. D **46**, 2355 (1992).
[2] V. A. Belinsky, I. M. Khalatnikov, and E. M. Lifshitz, Adv. Phys. **19**, 525 (1970); **31**, 639 (1982).
[3] A. A. Kirillov and A. A. Kochnev, Pis'ma Zh. Eksp. Teor. Fiz. **46**, 345 (1987) [JETP Lett. **46**, 435 (1987)].
[4] G. Montani, Tesi di Laurea, Universita di Roma, Facolta di Fisica, 1992.

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