Self-focusing and defocusing of electromagnetic-wave beams in the case of a thermal nonlinearity

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The self-effects of Laguerre electromagnetic beams of zeroth and first orders propagating in a weakly absorbing medium are analyzed. In a first-order perturbation theory in the nonlinearity, the axial intensity of the radiation is calculated for steady-state and unsteady-state propagation, for various focusing conditions. A generalization of the results to the case of a pronounced nonlinearity is proposed.

1. Introduction. The self-effects of wave beams of electromagnetic radiation in the optical and microwave ranges may differ qualitatively for different initial amplitudes and phase distributions of the beam field. For example, as a beam whose intensity falls off monotonically from the axis toward the periphery propagates through a defocusing medium, it experiences a defocusing, while regions with an intensity dip may result in a partial self-focusing of the beam.[1] These special features of the propagation of electromagnetic beams in nonlinear media are of both general physical interest and applied interest.

Mathematical difficulties confront efforts to solve a three-dimensional nonlinear boundary-value problem. Because of these difficulties, those studies which have been made of the self-effects of electromagnetic beams either have been based on overly simplified models or have used numerical simulation. In the latter case, the dependence of the beam characteristics on the various parameters of the problem found from the simulation by interpolation cannot be continued. Attempts to attack the problem within the framework of wave theory have been limited to the application of a perturbation theory directly to the wave equation, and they have proved successful in only a few particular problems with Gaussian beams.[2-4]

There may thus be some interest in the method proposed in this letter, which involves using a perturbation theory in a nonlinearity. This approach makes it possible to examine the most characteristic manifestations of self-effects of wave beams and to evaluate them quantitatively, by a common approach. In the present letter we are reporting the basic results of the use of this procedure for the examples of Laguerre beams of zeroth and first orders. The initial intensity distributions of these beams are a Gaussian distribution $\exp(-r^2)$ and a Gaussian distribution with an intensity dip at the axis, $r^4\exp(-r^2)$; the initial phase distribution corresponds to focusing at a certain distance.

2. General relations. Electromagnetic beams in a nonlinear medium are described by a nonlinear parabolic equation for the complex amplitude of the electromagnetic field:

$$2ik\frac{\partial A}{\partial z}+\Delta_\perp A+(2k^2n_1/n_0)A=0, \qquad (1)$$

where $n_1=n-n_0$ is the deviation of the refractive index of the medium from its unperturbed value n_0, $k=2\pi n_0/\lambda$ is the wave vector of a wave in the unperturbed medium, and z is the coordinate along the beam propagation direction. Introducing the dimensionless variables

$$x'=x/r_0,\quad y'=y/r_0,\quad z'=z/kr_0^2,\quad W=A/A_0,\quad \alpha'=\alpha kr_0^2$$

(we drop the primes below), where r_0 is a transverse scale length of the beam, $\alpha=4\pi\kappa/\lambda$ is the absorption index for the radiation in the medium, κ is the imaginary part of the refractive index of the medium, and A_0 is a typical initial value (at $z=0$) of the complex amplitude of the electromagnetic field, we can put Eq. (1) in the form

$$2i\frac{\partial W}{\partial z}+\Delta_\perp W+H(W)W=0\,, \qquad (2)$$

where $H(W)=2(kr_0)^2n_1(W)/n_0$. The radiation intensity $I(x,y,z)$ can be expressed in terms of the dimensionless function $W(x,y,z)$ as follows:

$$I(x,y,z)=(P_0/\pi r_0^2)\,|W|^2\exp(-\alpha z)\,,$$

where P_0 is the total beam power, and the function W is normalized by

$$\int\int_{-\infty}^{+\infty} |W(x,y,0)|^2 dxdy=\pi .$$

In order to derive a perturbation theory which makes it possible to find solutions of Eq. (2) for beams with arbitrary initial amplitude and phase distributions in the form of a series in the nonlinearity, we transform Eq. (2) into an integral equation, taking Fourier transforms along the transverse coordinates, $(x,y)\rightarrow(\rho,\eta)$, and integrating along z. As a result, we find the following integral equation for the Fourier transform $W(\rho,\eta,z)$ of the function $W(x,y,z)$:

$$W(\rho,\eta,z)=W_0(\rho,\eta,0)\left\{-\frac{i}{2}(\rho^2+\eta^2)z+i\int_0^z dz_1\frac{H(\rho,\eta,z_1,W)}{W(\rho,\eta,z_1)}\right\}. \tag{3}$$

Here $H(\rho,\eta,z,W)$ is the Fourier transform of the expression $H(W)W/2$, and the function $W_0(\rho,\eta,0)$ is the Fourier transform of the initial amplitude and phase distribution of the beam field. At this point it is convenient to use the Picard method of successive approximations, in which the solution of Eq. (3) for a wave beam propagating through an unperturbed medium is chosen as the zeroth approximation:

$$W_{\rm lin}(\rho,\eta,z)=W_0(\rho,\eta,0)\exp\left[-\frac{i}{2}(\rho^2+\eta^2)z\right].$$

An important point is that the zeroth-approximation functions incorporate the initial and boundary conditions of the problem. As a result, this procedure for constructing solutions of Eq. (3) is of universal applicability to beams with various initial amplitude and phase distributions. The first approximation, $W_1(x,y,z)$, is obtained by substituting $W_{\rm lin}(x,y,z)$ into the right side of (3); the second approximation is obtained by substituting in $W_1(x,y,z)$ (and so forth).

For the first approximation, $W_1(x,y,z)$, we find the expression

$$W_1(x,y,z)=W_{\rm lin}(x,y,z)+\frac{(kr_0)^2}{2\pi n_0}\int_0^z\frac{dz_1}{z-z_1}\int\int_{-\infty}^{+\infty}dx_1dy_1$$

$$\times\exp\left[-\frac{(x-x_1)^2+(y-y_1)^2}{2i(z-z_1)}\right]n_1W_{\rm lin},$$

which has the same form as the first approximation of the method of smooth perturbations. This is essentially a generalization of that method to the case of beams propagating in a nonlinear medium. Using $W_1(x,y,z)$, we find the following general relation for the relative intensity of a beam with an arbitrary initial amplitude and phase distribution in first-order perturbation theory:

$$I_{\rm rel}=I(x,y,z)/I_{\rm lin}(x,y,z)=1-N, \tag{4}$$

where

$$N=-\frac{(kr_0)^2}{2\pi n_0}{\rm Re}\left\{W_{\rm lin}^{-1}\int_0^z\frac{dz_1}{z-z_1}\int\int_{-\infty}^{+\infty}dx_1dy_1\right.$$

$$\times \exp\left[-\frac{(x-x_1)^2+(y-y_1)^2}{2i(z-z_1)}\right] n_1(x_1,y_1,z_1,W_{\rm lin}) W_{\rm lin}(x_1,y_1,z_1)\bigg\} .$$

Let us use this relation to calculation the intensities of Laguerre beams in a medium whose nonlinearity results from a heating by the energy of the electromagnetic field.

3. An L_0 beam, i.e., a Gaussian beam. We restrict the discussion to the case of a long radiation pulse, for which the condition $t \gg r_0/c_8 F$ holds, where c_s is the sound velocity in the medium, $F=kr_0^2/z_F$ is the Fresnel number, and z_F is the focal length. We assume that the primary mechanism for heat transfer out of the volume occupied by the electromagnetic field is the drift of the heated medium out of the propagation channel (at a velocity $v_\perp$ along the x axis). We accordingly consider two beam propagation regimes: an unsteady regime ($t \ll t_1 = r_0/v_\perp F$), in which the rate of heat removal is much lower than the heating rate, and a steady state ($t \gg t_1$), in which the heating rate and the rate of heat removal are comparable. In the unsteady regime, the change in the refractive index of the medium is determined by the expression

$$n_1 = -\frac{(n_0-1)(\gamma-1)\alpha I_{\rm lin} t}{\gamma p_0},$$

while in the steady state it is determined by

$$n_1 = -\frac{(n_0-1)(\gamma-1)r_0}{\gamma p_0 v_\perp}\int_{-\infty}^{x} I_{\rm lin}(x_1,y,z)\,dx_1 ,$$

where p_0 and γ are the pressure and the ratio of specific heats of the medium.

The relative intensity calculated from (4) for a Gaussian beam in an unsteady regime in a weakly absorbing medium ($\tau = \alpha z \ll 1$), is

$$I_{\rm rel}(z) = 1 - \frac{t}{2\sqrt{\pi}t_{\rm cr} z^2}\ln\frac{(1-zF)^2+9z^2}{(1-zF)^2+z^2}, \qquad (5)$$

where we have introduced the nonlinearity parameter (in dimensional units)

$$\frac{1}{t_{\rm cr}} = \frac{(n_0-1)(\gamma-1)\alpha P_0 z^2}{2\sqrt{\pi} n_0 \gamma p_0 r_0^4}.$$

In the limiting cases of a collimated beam ($F=0$) and a focused one ($z=1/F$), expression (5) yields[2,3]

$$I_{\rm rel}(z) = 1 - 4t/\sqrt{\pi}t_{\rm cr}, \quad z \ll 1 ,$$

$$I_{\rm rel}(z_F) = 1 - tF^2 \ln 3/\sqrt{\pi}t_{\rm cr} .$$

A calculation of the relative intensity of a Gaussian beam in a steady state from (4), for a collimated beam, yields the known result[4]

$$I_{\rm rel}(z) = 1 - N_0\Phi(\tau), \quad z \ll 1,$$

where $N_0=r_0/v_\perp\, t_{cr}$ and $\Phi(\tau)=2[\tau-1+\exp(-\tau)]/\tau^2$. For a focused beam we find, at the focal point,

$$I_{rel}(z_F)=1-N_0G_0(F,\tau)\ , \tag{6}$$

where

$$G_0(F,\tau)=\frac{\pi F^2}{3\sqrt{1+3F^2}}\left[1-\frac{6F}{\pi(1+3F^2)}\ln\frac{1+3F^2}{eF}\right]\exp\left(-\frac{\tau F}{0.6+F}\right)$$

in the approximation which incorporates the main terms of an expansion in the parameter $2F/(1+3F^2)\leqslant 1/\sqrt{3}$. This result, derived in wave theory, leads to a critical power for thermal defocusing which is about three times the result found through a geometric-optics calculation.[6]

4. An L_1 beam, i.e., a first-order Laguerre beam. An L_1 beam is described in an unperturbed medium by the following solution of Eq. (2):

$$W_{lin}=\frac{i\sqrt{2}z}{[1+iz(1+iF)]^2}\exp\left(-\frac{(x^2+y^2)(1+iF)}{2[1+iz(1+iF)]}\right)L_1\left(\frac{i(x^2+y^2)}{2z[1+iz(1+iF)]}\right),$$

where L_1 is the Laguerre first-degree polynomial, $L_1(x)=1-x$. The self-effect of an L_1 beam is qualitatively different from that of a Gaussian beam. In an unsteady regime, a collimated beam is focused at distances $z\ll 1$,

$$I_{rel}(z)=1+\frac{376}{15}\frac{4tz^2}{\sqrt{\pi t_{cr}}}, \tag{7}$$

while a focused beam is defocused at the focal point,

$$I_{rel}(z_F)=1-\frac{tF^2\ln 3}{\sqrt{\pi t_{cr}}}q_{nst}\ , \tag{8}$$

where $q_{nst}=15/16-1/108\ln 3\approx 0.93$. For steady-state beam propagation, we find corresponding effects: an underfocusing

$$I_{rel}(z)=1+(5/24)N_0 \tag{9}$$

for a collimated beam at distances $z\ll 1$ and a weakened defocusing

$$I_{rel}(z_F)=1-N_0G_1(F,\tau) \tag{10}$$

for a focused beam. Here, under the condition $F\gg 1$, we have

$$G_1(F,\tau)=\frac{\pi F}{3\sqrt{3}}\exp(-\tau)q_{st}\ ,\ q_{st}=\frac{35}{48}\left[1+\frac{19}{21}\left(1-\frac{9\sqrt{3}}{5\pi}\right)\right]\approx 0.73$$

We used the condition $\tau\ll 1$ in deriving (7)–(9).

5. Generalized formulas. For each type of beam considered here, the expressions derived above for the intensity can be written in the form of common formulas, which apply to both the unsteady and steady regime. In the limiting cases of short times ($t\ll r_0/v_\perp F$) and long times ($t\gg r_0/v_\perp F$), these common formulas become the corresponding limiting cases. It is also interesting to generalize these results for defocusing beams to the case of a strong nonlinearity, since an increase in the nonlinearity parameter (due to an increase in the beam power, for example) is accompanied by an increase in the beam intensity in the focal plane only up to a certain maximum value. If the power is raised further, this intensity decreases. Some approximate expressions for the behavior of the axial intensity of a beam as a function of its power were proposed in Refs. 5–7 on the basis of the results of numerical simulations. In the region of a weak nonlinearity, however, those expressions become a power law P_0^α, where $\alpha \neq 1$, in contradiction of the physical meaning and also in contradiction of the results calculated by perturbation theory. The generalization which we are proposing here is to replace the expression $1-N$ in (4) by $\exp(-N)$. This generalization leaves the results found by first-order perturbation theory in the nonlinearity valid, and it gives a qualitatively correct description of the behavior of the intensity of a defocused beam in the case of a pronounced nonlinearity. At the same time, it allows us to estimate the critical parameters of the beam.

As an example, we write a generalized formula of this sort for a focused Gaussian beam and a focused L_1 beam under the assumptions $F \gg 1$ and $\tau \ll 1$:

$$I_{\text{rel}}(z_F) = \exp\left(-\frac{q_{\text{nst}} t F^2 \ln 3}{\sqrt{\pi} t_{\text{cr}}} \left[1 + \left(\frac{3}{\pi} \right)^{3/2} \frac{q_{\text{nst}} \ln 3}{q_{\text{st}}} \frac{t v_\perp}{r_0} F \right]^{-1} \right), \tag{11}$$

where $q_{\text{nst}} = q_{\text{st}} = 1$ for an L_0 beam, and $q_{\text{nst}} = 0.93$ and $q_{\text{st}} = 0.73$ for an L_1 beam. In particular, we see from (11) how the "fictitious" singularity along $v_\perp$ in the equations for the steady-state propagation regime is eliminated by the transition to the unsteady regime.

[1)]An alternative English transliteration of the author's name is M. A. Fyodorov.

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