# Pattern recognition through a Derrida model with rare bonds

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A Derrida model with rare bonds is discussed for pattern recognition. The critical values of the bondedness and the number of patterns required for pattern recognition are found.

The Derrida model is a very simple idea of the spin-glass state. An exact solution can be derived for this model with even a single deviation from replica symmetry in the Parisi scheme.[2] This system thus seems to have several unique properties. For example, it offers the opportunity for an optimum coding of information in the presence of noise.[3]

Gardner[4] has used a complete-bonding version of this model for pattern recognition. In the present letter we pursue the ideas of Ref. 4, for the case of rare bonds. We use the technique of Ref. 5 for models with rare bonds. We assume that we are given $M$ random patterns

$$\{\xi_c^\mu\},\quad (\mu=1...M,\ i=1...N),\quad \xi_i^\mu=\pm 1.$$

We wish to construct a Hamiltonian $H$ for $N$ spins $\sigma_i$ whose unique ground state would be $M$ sets of spins $\{\xi_i^\mu\}$. We consider

$$H=-\sum_{(i_1...i_p)} \sigma_{i_1}...\sigma_{i_p} j_{i_1..i_p}, \tag{1}$$

where

$$j_{i_1...i_p}=\sum_{\mu=1}^{M} \xi_{i_1}^\mu...\xi_{i_p}^\mu. \tag{2}$$

The summation in (1) is not over all $C_N^p$ possible choices of the set of different indices $(i_1...i_p)$ but over only $Z=cN$ of these choices, selected at random. In the case at hand we choose $1\ll p\ll N$ and $C_N^p\approx N^p/p!$. Hamiltonian (1) is used in the Derrida model, but the interaction constants $j_{i_1...i_p}$ have a Gaussian[1] or discrete[5] distribution. We can assume that the summation in (1) is over all sets of indices $(i_1...i_p)$ without limitation, but now the constants $j_{i_1...i_p}$ have the distribution

$$\rho(j_{i_1...i_p})=\left(1-\frac{cp!}{N^{p-1}}\right)\delta_{j_{i_1...ip},0}+\frac{cp!}{N^{p-1}}\delta_{j_{i_1...i_p}},\ \left(\sum_{\mu=1}^{M} \xi_{i_1}^\mu...\xi_{i_p}^\mu\right). \tag{3}$$

We will use the latter representation in calculating the free energy of the model by the mean-field method.

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By virtue of the gauge invariance of the system, instead of the patterns $\{\xi_i^\mu\}$ we can deal with the new patterns in (1) and $\{\xi_i^1,\xi_i^\mu,\ \mu\geqslant 2$, for which Eq. (2) becomes

$$j_{i_1\ldots i_p}=1+\sum_{\mu=2}^{\mu}(\xi_{i_1}^1\xi_{i_1}^\mu)\ldots(\xi_{i_p}^1\xi_{i_p}^\mu). \tag{4}$$

In discussing a ferromagnetic phase we can use representation (4). We use the formula

$$e^{jB\sum_{\alpha=1}^{n}\tau\sigma_\alpha}=(\cosh Bj)^n\left[1+\sum_{r=1}^{\infty}(\tanh Bj)^r\sum_{(\alpha_1\ldots\alpha_r)}\sigma_{\alpha_1}\ldots\sigma_{\alpha_r}\tau^r\right], \tag{5}$$

where the summation on the right side is over all possible sets $(\alpha_1\ldots\alpha_r)$ of the various indices. Considering $n$ copies of spins with Hamiltonian (1), we find the following expression for the free energy (see Ref. 5 for a derivation):

$$-nBF=cn<\ln(\cosh Bj)>-\sum_{r=1}^{\infty}\sum_{\alpha_1\ldots\alpha_2}\lambda_{\alpha_1\ldots\alpha_r}Q_{\alpha_1\ldots\alpha_r}$$
$$+c\sum_{r=1}^{\infty}<[\tanh Bj]^r>\sum_{\alpha_1\ldots\alpha_r}(Q_{\alpha_1\ldots\alpha_r})^p+\ln \mathrm{Tr}\exp\left[\sum_{r=1}^{\infty}\lambda_{\alpha_1\ldots\alpha_r}\sigma_{\alpha_1}\ldots\sigma_{\alpha_r}\right], \tag{6}$$

where $\lambda_{\alpha_1\ldots\alpha_2}$ are Lagrange multipliers and $Q_{\alpha_1\ldots\alpha_r}$ are spin correlation functions. The random quantity is defined as the sum of $M$ random numbers $\xi_i^\mu=\pm 1$:

$$j=\sum_{\mu=1}^{M}\xi^\mu, \tag{7}$$

where $\xi^\mu$ has the same distribution as $\xi_i^\mu$ at a fixed $i$.

Below we consider the case in which the quantities $\xi_i^\mu$ are distributed uniformly. The results can easily be generalized to the case of a nonuniform distribution of $\xi_i^\mu$. As in (4), in examining the ferromagnetic phase we start from

$$j=1+\sum_{\mu=2}^{M}\xi^\mu, \tag{8}$$

where $\xi^\mu$ are distributed uniformly.

We first consider solutions of (6) with replica symmetry conserved. We have

$$\lambda_{\alpha_1\ldots\alpha_r}=\lambda_r,\quad Q_{\alpha_1\ldots\alpha_r}=q_r.$$

From (6) and from the condition for an extremum of the free energy with respect to $q_r$ we find

$$\lambda_r=cp<(\tanh Bj)^r>q_r^{p-1}. \tag{9}$$

Since we have $q_r\leqslant 1$, the following cases are possible:

$$q_r=1,\quad \lambda_r\to\infty \tag{10}$$

and

$$q_r < 1, \quad \lambda_r \to 0. \tag{11}$$

The paramagnetic phase corresponds to choice (11). Expanding $\ln \mathrm{Tr}_\sigma \exp \Sigma_r \lambda_2 \times \Sigma_{\alpha_1 .. \alpha_r} \sigma_{\alpha_1} ... \sigma_{\alpha_r}$ in a series, we find that terms with $(\lambda_r)^2$ and higher are nonvanishing. Differentiating (6) with respect to $\lambda_r$, we thus find $q_r = 0$. For the free energy of the paramagnetic phase we then find

$$-BF = c < \ln(\cosh Bj) > + \ln 2\,. \tag{12}$$

We turn now to the ferromagnetic phase. We assume that $q_1$ is nonzero (this is the case of ordinary magnetization). If we had $q_r < 1$ we would have again found the case of paramagnetism from (11). If we instead set $q_r = 1$, we find $\lambda_r \to \infty$ from (10). Within small terms like $\exp(-\lambda_1)$ we then find

$$\ln \mathrm{Tr}_\sigma \exp\left( \sum_r \lambda_r \sum_{\alpha_1 ... \alpha_r} \sigma_{\alpha_1} ... \sigma_{\alpha_r} \right) = \sum_r \lambda_r C_n^r). \tag{13}$$

Using (13) in (6), and working from the condition for an extremum with respect to $\lambda_r$, we find that for all $r$ we have

$$q_r = 1 \tag{14}$$

and, correspondingly,

$$\lambda_r \to \infty\,. \tag{15}$$

Substituting (14) and (15) into (6), and taking the limit $n \to 0$, we find

$$-nBF = cn \sum_{r=1}^{\infty} < \ln(\cosh Bj) > + c \sum_{r=1}^{\infty} < (\tanh Bj)^r > C_n^r$$

$$= cn < \ln(\cosh Bj) > + c < [1 + \tanh(Bj)]^n - 1 > = cnB. \tag{16}$$

We have used representation (8) here. In principle, there might be (some purely hypothetical) solutions with $q_1 = 1$, $q_r = 1$ for certain values of $r$, but we see no physical basis for such solutions in the phase of replica-symmetry conservation. We now move on to the spin-glass phase. It turns out that for odd values of $r$ we have $q_r = 0$ (and we are back in the phase of ferromagnetism). We are thus dealing with absolutely the same case as in Ref. 5, where a model with a random symmetric distribution of the constants $j_{i_1 ... i_p}$ was examined. As was shown in Ref. 5, an exact solution for the spin-glass phase can be found even if there is a single deviation from replica symmetry (an examination of higher orders of the disruption of replica symmetry reveals no change in the solution). The free energy can be found from the corresponding solution for the paramagnetic phase, through an examination of some critical $B_c$, at which the entropy disappears.

We thus have the equation

$$c < \ln(\cosh B_c j) > + \ln 2 - c < j \tanh(B_c j) > B_c = 0. \tag{17}$$

At and above this temperature, the free energy does not change. We thus have the following for this case (in the spin-glass phase):

$$-B_c F \equiv c < \ln(\cosh B_c j) > + \ln 2 = c B_c < j \tanh(B_c j) > . \quad (18)$$

We have found three phases for our system. In contrast with the case of a finite $p$, there are no phases with a partial magnetization. The situation regarding the phases in the case of complete bonding[4] is unchanged in our case of rare bonds. At which values of the parameters ($c$, $M$, and $B$) do transitions occur between these phases? To be mathematically rigorous, we would have to study the stability of the solutions found here with respect to perturbations. In our situation, with random bonds, however, such a study would be very difficult (if possible at all). On the other hand, it is clear that all three of these phases should exist, at least for appropriate values of the parameters.

A ferromagnetic phase exists under the conditions $c \gg 1$ and $B \gg 1$. A small paramagnetic phase exists under the condition $B \ll 1$, and there is a spin glass under the conditions $B \gg 1$, $c \gg 1$, and $M \gg 1$. Our only remaining problem is to find the phase boundaries. To do this we work from the condition that the free energies are equal.

When does the transition from the phase of ferromagnetism to the spin-glass phase occur? From the condition for the joining of solutions (16) and (18) we find

$$1 = < j \tanh(B_c j) > . \quad (19)$$

For $M=2$ we find from (19)

$$1 = \tanh(2B_c). \quad (20)$$

We thus have $B_c \to \infty$, and we find the following critical value of $c$ from (17):

$$c = 2. \quad (21)$$

We thus need $2N$ bonds in order to record two patterns. In the case $M > 2$, we would instead need $Z > MN$ bonds.

When we rewrite the distribution of $j$ in (8) in the form

$$\rho(j) = \sum_{k=-M}^{M} \rho_k \delta_{k,j}, \quad (22)$$

Eqs. (19) and (17) become

$$1 = \sum_{k=-M}^{M} \rho_k \tanh(kB_c), \quad (23)$$

$$c = \frac{\ln 2}{\sum_{k=-M}^{M} \rho_k [kB_c \tanh(B_c k) - \ln(\cosh kB_c)]} . \quad (24)$$

Finding $B_c$ from (23), and substituting the result into (24), we find the critical value of $c$. In the limit $M \to \infty$, Eqs. (23) and (24) simplify, and we find

$$B_c = 1/M, \quad c = M \ln 2. \quad (25)$$

We see that the Hamiltonian in (1) does not record the patterns in the optimum way. It would be very interesting to find a Hamiltonian which would record $M$ patterns in the optimum way, as in the coding case.

We now consider Hamiltonian (1) in the case of complete bonding. A curious effect exists in this case; this effect was actually seen in Ref. 4.

The concept of an expenditure of free energy for the selection of the vacuum during phase transitions in statistical physics was introduced in Ref. 6. Denoting by $Z(N,T)$ the partition function of the model on a lattice with $N$ nodes at the temperature $T$, and assuming that a phase transition occurs at some critical temperature, we conclude that there is an abrupt change in the asymptotic free energy. If an asymptotic expansion

$$F(N,T) \equiv \ln Z(N,T) = Nf_0(T) + ... f_1(T) + ... O(1) \tag{26}$$

exists, then we have

$$\lim_{e\to 0} f_1(T_c - \epsilon) - f_1(T_c + \epsilon) = \text{const} \ln Q\,. \tag{27}$$

The existence of an abrupt change in (27) has been recognized for a long time now on the basis of the exact solution of the Ising model on a finite square lattice. In this case we have const$=1$, and the order of the symmetry group is $Q=2$.

Numerical calculations[6] have shown that in the case of lattices with the topology of a torus we have const$=1$ in (27) for the $d2$, $Q=3$, and $Q=5$ Potts models and also for the $d3$ Ising model, as for a model with a Lee–Yang singularity on a random dynamic lattice. For the $d2$ Ising model on a random dynamic lattice of kind $g$ we have const$=g$. In our case, for $M$ patterns, we find

$$\lim_{\epsilon\to 0} f_1(T_c - \epsilon) - f_1(T_c + \epsilon) = \ln M. \tag{28}$$

In order to choose the appropriate pattern among the $M$ possible, the system increases its entropy by $\ln M$. In our case the Hamiltonian does not have an exact symmetry group [the energies of the various vacua differ by $O(1)$], in contrast with the models of Refs. 6–8. The result in (27) apparently has the property of universality.

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