Path-integral quantization of the straight-line string

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A path-integral quantization of the relativistic straight-line string is proposed. In an explicitly covariant form starting with the initial four-dimensional dynamics of the relative coordinate r_μ the three-dimensional one is derived. A connection between constraints which appear in the canonical formalism and the path integral quantization is discussed.

1. Introduction. Various kinds of string-like models have been employed to describe the world of hadrons.[1] The massless relativistic straight-line string is usually considered as the simplest dynamical basis of the model of hadrons. This model has been quantized in the canonical quantization formalism.[2] Dynamics of quark and gluon fields leads us to the study of dynamics of the relativistic string with masses at the ends,[3] in which the quarks carry a finite fraction of the energy-momentum of the hadron. But even the simplest version of the straight-line string with masses at the ends cannot be made tractable in the canonical quantization formalism.[4] In this paper we propose a path-integral Lorentz-covariant approach to the quantization of the massless relativistic straight-line string which can be generalized to the case of the straight-line string with masses at the ends.[6]

2. Gaussian representation for the action of the straight-line string. The standard form of the action of the straight-line string in Euclidean space is

$$S=\sigma_0\int_0^1 d\gamma\int_0^1 d\beta\ [\dot{w}^2\times w'^2-(\dot{w}\times w')]^{1/2}, \tag{1}$$

where $w_\mu(\gamma,\beta)$ are the coordinates of the string world surface

$$w_\mu(\gamma,\beta)=z_\mu(\gamma)\times\beta+\bar{z}_\mu(\gamma)\times(1-\beta), \tag{2}$$

and $z_\mu(\gamma)$, $\bar{z}_\mu(\gamma)$ are the coordinates of the ends of the string. The dot and the prime stand for the derivatives over γ and β parameters. Therefore we are to consider and make tractable the following functional integral in the Euclidean space:

$$G=\int Dz_\mu(\gamma)D\bar{z}_\mu(\gamma)\exp[-S]\ . \tag{3}$$

The action is invariant under the reparametrization

$$\gamma\to f(\gamma,\beta),\quad w_\mu(\gamma,\beta)\to w_\mu[f(\gamma,\beta),\beta]\ , \tag{4}$$

where the function $f(\gamma,\beta)$ satisfies the conditions

$$f(0,\beta)=0,\ f(1,\beta)=1,\frac{\partial f(\gamma,\beta)}{\partial\gamma}>0\ . \tag{5}$$

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It is convenient to introduce a "center-of-mass" coordinate $R_\mu(\gamma)$ and a relative coordinate $r_\mu(\gamma)$ as follows:[3]

$$R_\mu(\gamma)=\tfrac{1}{2}[\bar{z}_\mu(\gamma)+z_\mu(\gamma)],\ r_\mu(\gamma)=z_\mu(\gamma)-\bar{z}_\mu(\gamma), \tag{6}$$

so that

$$w_\mu=R_\mu+(\beta-1/2)\times r_\mu\,. \tag{7}$$

The boundary conditions can be imposed in the Lorentz-invariant way:

$$R_\mu(1)-R_\mu(0)=T\times u_\mu,\ u_\mu\times u^\mu=1, \tag{8}$$

and $r_\mu(0)$, $r_\mu(1)$ are also fixed.

To develop a procedure to evaluate path integral (3), we use the auxiliary fields formalism, as is usually done in the string theory.[5]

Let us rewrite (3) as follows:

$$\begin{aligned}G&=\int Dr\,DR\,Dh_{ab}\exp\left[-\sigma_0\int\sqrt{h}d^2\xi\right]\delta[\partial_a w_\mu\partial_b w^\mu-h_{ab}(\xi)]\\&=\int Dr\,DR\,Dh_{ab}\int_{-i\infty}^{+i\infty}D\lambda^{ab}\exp\left[-\sigma_0\int\sqrt{h}d^2\xi\right]\\&\quad\times\exp[+\sqrt{h}\lambda^{ab}h_{ab}d^2\xi]\exp\left[-\int\sqrt{h}\lambda^{ab}\partial_a w_\mu\partial_b w^\mu d^2\xi\right],\end{aligned} \tag{9}$$

where $d^2\xi=d\gamma d\beta$, $\xi_1=\gamma$, $\xi_2=\beta$, and $\equiv\det h$.

It is convenient to decompose[5]

$$\lambda^{ab}(\xi)=\alpha(\xi)h^{ab}(\xi)+f^{ab}(\xi), \tag{10}$$

with

$$f^{ab}h_{ab}=0,\ h^{ab}\equiv(h^{-1})^{ab}. \tag{11}$$

In a similar way as was done in the case of Nambu–Goto string,[5] we can prove that in the continuum limit $\alpha(\xi)$ and $f^{ab}(\xi)$ can be replaced by their mean values

$$\langle\alpha(\xi)\rangle\to\bar{\alpha},\quad\langle f^{ab}(\xi)\rangle\to 0\,. \tag{12}$$

Equation (12) reflects the fact that $\alpha(\xi)$ is a scalar, while $f^{ab}(\xi)$ is a traceless tensor.

Using (12), we obtain the following expression for G:

$$G=\int Dr\,DR\,Dh_{ab}\,\exp\left[-(\sigma_0-2\bar{\alpha})\int\sqrt{h}d^2\xi\right]\exp\left[-\bar{\alpha}\int\sqrt{h}h^{ab}\partial_a w_\mu\partial_b w^\mu d^2\xi\right], \tag{13}$$

with the new action which is quadratic in w_μ and which contains the new auxiliary fields h_{ab}.

3. Integration over the auxiliary fields. The invariance (4) makes it convenient to introduce the new variables $\bar{v}(\beta)$, $f(\xi)$, and $\eta(\xi)$. Separating out the collective mode $\bar{v}(\beta)$ and the field $f(\gamma,\beta)$, which satisfies conditions (5), we have

$$\hbar\equiv\frac{h}{h_{22}^2}+[T\sigma\bar{v}(\beta)]^2\left(\frac{\partial f(\gamma,\beta)}{\partial\gamma}\right)^2. \tag{14}$$

Making a simple rescaling of $\hbar_{12}$, we also introduce the variable $\eta(\gamma,\beta)$ instead of h_{12}

$$\hbar_{12}\equiv\frac{h_{12}}{h_{22}}=\left(\frac{\partial f(\gamma,\beta)}{\partial\gamma}\right)[T\eta(\gamma,\beta)], \tag{15}$$

where T is contained in boundary condition (8). Taking into account the fact that

$$Dh_{11}\ Dh_{22}\ Dh_{12}=D\hbar\ D\hbar_{12}\ h_{22}^2\ Dh_{22}, \tag{16}$$

and using the well-known formula in the string theory[5]

$$D\hbar\sim\exp\left[-\frac{\text{const}}{\epsilon}\int\sqrt{\hbar}d^2\xi\right]D\bar{v}(\beta)\,Df(\gamma,\beta), \tag{17}$$

where $1/\epsilon\sim\Lambda$ is the ultraviolet cutoff scale, we obtain the following expression after replacing the integration over $d\gamma$ by $Tdf(\gamma,\beta)\equiv d\tau$ and Gaussian integration over $h_{22}\geqslant 0$

$$G=\int DR_\mu\ Dr_\mu D\bar{v}(\tau,\beta)d\eta(\tau,\beta)\exp[-A], \tag{18}$$

where

$$A=\int_0^T d\tau\int_0^1 d\beta\frac{1}{2\bar{v}}[\dot{w}^2+(\sigma\bar{v})^2r^2-2\eta(\dot{w}r)+\eta^2r^2], \tag{19}$$

and the trivial rescaling $z,\bar{z}\rightarrow(\sigma/2\alpha)^{1/2}\ z,(\sigma/2\alpha)^{1/2}\ \bar{z}$ has been done.

At first, we notice that the action does not depend on $f(\gamma,\beta)$ which reflects the invariance (4). The integral over $Df(\tau,\beta)$ can therefore be factored out and it is equal to the volume of the reparametrization group.

In the standard way[5] we have introduced the physical quantity σ, which is contained in expressions (14) and (19)

$$\sigma^2=\bar{\alpha}\left(\sigma_0-2\bar{\alpha}+\frac{\text{const}}{\epsilon}\right). \tag{20}$$

In the action the function $\eta(\tau,\beta)$ is integrated over β which is multiplied by the function $\bar{v}(\beta)$. In what follows the integration over $\bar{v}$ is performed by the steepest-descent method and in the extremum $\bar{v}[(\beta-\frac{1}{2})]=\bar{v}[-(\beta-\frac{1}{2})]$. We can therefore consider only the class of functions $\bar{v}[(\beta-1/2)^2]$, which are even functions of $(\beta-1/2)$.

It is convenient to decompose the functions $\eta(\tau,\beta)$ in orthogonal polynomials $P_n(\beta)$ with the weight $v(\beta)\equiv 1/\bar{v}(\beta)$

$$\eta(\tau,\beta)=\sum_n P_n(\beta)k_n(\tau), \tag{21}$$

$$\int_0^1 d\beta\nu(\beta)P_n(\beta)P_m(\beta)=\delta_{mn}\,. \tag{22}$$

After a Gaussian integration over $R(\gamma)$ with the condition (8)

$$\int DR\rightarrow\int D\dot{R}\int_{-i\infty}^{i\infty}d^4\lambda\exp\left[\int_0^T\lambda^\mu(\dot{R}_\mu-u_\mu)d\tau\right], \tag{23}$$

it is easy to prove that the action can be represented, with λ being rewritten as $i\lambda$, in the form

$$S=\frac{1}{2}\int_0^T d\tau\left[a_3\,\dot{r}^2+\sigma^2a_{-1}r^2+r^2\sum_{n=1}^{\infty}k_n^2(\tau)-2i(a_1)^{-1/2}k_0(\tau)(\lambda r)\right.$$
$$\left.-2(a_3)^{1/2}(\dot{r}r)k_1(\tau)+\frac{\lambda^2}{a_1}+2i(\lambda u)\right], \tag{24}$$

where we have introduced the notation

$$a_1=\int_0^1\nu d\beta,\quad a_3=\int_0^1(\beta-1/2)^2\nu d\beta,\ a_{-1}=\int_0^1\frac{d\beta}{\nu}\,. \tag{25}$$

The function $k_0(\tau)$ enters only the fourth term in this expression and the integration over $Dk_0(\tau)=\Pi_{i=1}^N dk_0(\tau_i)$, where N tends to infinity, gives the factor which is proportional to the infinite product of δ-functions;

$$\prod_{i=1}^{\infty}\delta[\lambda r(\tau_i)]. \tag{26}$$

This means that there is a dynamical condition

$$(\lambda r)=0. \tag{27}$$

Integrations over $d^4\lambda$ and Dk_n with $n\geqslant1$ lead (effectively in the limit $T\rightarrow\infty$) to the following expression (up to a change of the measure):

$$G=\int d\nu(\beta)\,Dr_\mu\delta(ru)\exp\left\{-\frac{1}{2}\int_0^T d\tau\left[a_1+\left(\dot{r}^2-\frac{(r\dot{r})}{r^2}\right)a_3+\sigma^2a_{-1}\ r^2\right]\right\}, \tag{28}$$

where we have used the fact that the extremum value of λ_μ is

$$\lambda_\mu\sim u_\mu\,. \tag{29}$$

It is important that there are two constraints

$$(ru)\sim(rP)=0, \tag{30}$$

$$(rp)=0, \tag{31}$$

where P_μ is the total momentum of the string, and

$$p_\mu = a_3\left(\dot{r}_\mu - \frac{(\dot{r}r)r_\mu}{r^2}\right) \tag{32}$$

is the relative momentum of the string.

The first constraint (30) means that only the components of r_μ, transverse to the total momentum P_μ, are responsible for the dynamics of the string. The second one (31) reflects the fact that the action does not depend on the components of p_μ which are longitudinal to r_μ.

Let us consider the rest system of the meson $u_\mu = (1, \bar{0})$ and go over from the Euclidean space to the Minkowski space

$$d\tau_E \to i d\tau_M. \tag{33}$$

The Hamiltonian corresponding to the action (28) is

$$H(\mathbf{p},\mathbf{r}) = 1/2\left\{\frac{1}{a_3}\frac{\hat{L}^2}{\mathbf{r}^2} + \sigma^2 a_{-1}\mathbf{r}^2 + a_1\right\}, \tag{34}$$

where $\mathbf{L} = (\mathbf{r} \times \mathbf{p})$ is the angular momentum operator.

Since the Hamiltonian does not contain the radial part of the kinetic term, the field $\mathbf{r}^2$ is not a dynamic field. In the spirit of the canonical formalism we must therefore exclude the field $\mathbf{r}^2$. This can be done by solving its equation of motion for a fixed value of the orbital momentum:

$$-\frac{l(l+1)}{a_3\mathbf{r}^4} + \sigma^2 \times a_{-1} = 0. \tag{35}$$

Inserting this extremum value of $\mathbf{r}^2$ into expression (34), we obtain the final expression for the Hamiltonian

$$H(\nu,l) = \frac{1}{2}a_1 + \sigma\sqrt{a_{-1}/a_3}\sqrt{l(l+1)}\,. \tag{36}$$

Solving Eq. (36) for the extremum of $\nu(\beta)$ with the conditions (25), we find

$$\nu(\beta) = \left(\frac{8\sigma\sqrt{(l+1)l}}{\pi}\right)^{1/2} \frac{1}{\sqrt{1-4(\beta-1/2)^2}}, \tag{37}$$

where $\nu(\beta)$ plays the role of the energy density of the string. This solution corresponds to the spectrum of the Hamiltonian

$$E_l^2 = M_l^2 = 2\pi\sigma\sqrt{l(l+1)}, \tag{38}$$

which agrees with the result obtained for the straight-line string in the canonical quantization formalism.

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