# Level spacing distribution near the Anderson transition

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For a disordered system near the Anderson transition we show that the nearest-level-spacing distribution has the asymptotic behavior $P(s) \propto \exp(-As^{2-\gamma})$ for $s \gg \langle s \rangle \equiv 1$, which is universal and intermediate between the Gaussian asymptotics in a metal and the Poisson asymptotics in an insulator. (Here the critical exponent is in the range $0<\gamma<1$, and the numerical coefficient $A$ depends only on the dimensionality $d>2$.) It is obtained by mapping the energy level distribution onto the Gibbs distribution for a classical one-dimensional gas with a pairwise interaction. The interaction, which is consistent with the universal asymptotic behavior of the two-level correlation function found previously, was found to be the power-law repulsion with the exponent $-\gamma$.

It is well known,[1-3] that a statistical description of the energy levels of disordered quantum systems in the metallic phase is provided by the random matrix theory[4] (RMT). Its most important characteristic is the repulsion between the energy levels at any scale. In the Anderson insulator phase the energy levels are independent and are described by the Poisson statistics provided that the appropriate states are separated by a length exceeding the localization length.

It was assumed[5,6] that a universal statistical description is possible in the critical region near the Anderson transition between the two phases. The dimensional scaling estimated in Ref. 5 for the variance $\langle N^2 \rangle - \langle N \rangle^2$ of the number of energy levels in a given energy interval has suggested that is it proportional to $\langle N \rangle$, which makes it different from the Poisson statistics only by a certain number. A different statistical characteristic, the nearest-level-spacing distribution, was assumed in Ref. 6 on the basis of numerical simulations to be a universal "hybrid" of the Poisson distribution for large level spacings and the Wigner hypothesis (see below) for small level spacings.

However, it was recently proved analytically[7] that the universal statistics, which are exactly applicable near the Anderson transition point (the mobility edge), are entirely new and drastically different from both the RMT and the Poisson limit. The

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variance of the number of levels in the energy interval $E \gg \Delta$ (centered at the Fermi energy) was found to be

$$\langle N^2\rangle - \langle N\rangle^2 = \frac{a_d}{\beta}\left(\frac{E}{\Delta}\right)^{\gamma} \equiv \frac{a_d}{\beta}\langle N\rangle^{\gamma}, \quad 0<\gamma<1. \tag{1}$$

Here $\Delta$ is the mean level spacing, $\langle \cdots \rangle$ denotes the ensemble averaging, the coefficient $a_d$ and the critical exponent $\gamma$ depend only on the dimensionality $d>2$, and $\beta$ is determined by the class of symmetry ($\beta=1$, 2, or 4 for unitary, orthogonal, and symplectic ensembles, respectively[4]). Equation (1) is exact at the mobility edge in the limit

$$L\to\infty, \quad \langle N\rangle = \text{const} \gg 1. \tag{2}$$

In the same limit the metallic phase is described exactly by the RMT and the insulator phase is described by the Poisson statistics.

The spectral rigidity therefore does not disappear at the mobility edge (in contrast with the insulating phase, where the levels are independent and the variance is $\langle N\rangle$), but it is considerably weaker than in the metallic phase (where the fluctuations are suppressed and the variance is proportional to $\ln\langle N\rangle$). We can assume, therefore, that the nearest-level-spacing distribution near the Anderson transition is different from that in the metal phase and that in the insulating phase.

We will show in this letter that the asymptotic behavior of this distribution at the mobility edge is given by

$$P(s) \propto \exp(-A_d\beta\, s^{2-\gamma}), \quad s\equiv \omega/\Delta \gg 1, \tag{3}$$

where $\omega$ is the distance between the adjacent levels, and $A_d$ is a numerical factor which depends only on the dimensionality $d$. This case is drastically different from the Poisson distribution, $P(s)=\exp(-s)$, and from the exact Gaussian asymptotics in the metallic phase[4]

$$P(s) \sim \exp(-\tfrac{1}{16}\pi^2\beta\, s^2). \tag{4}$$

Note that the Wigner hypothesis, $P(s)=(\pi s/2)\exp(-\pi s^2/4)$ (for $\beta=1$), describes the asymptotic behavior only approximately.[4]

Both the universal variance (1) and the asymptotic behavior of the distribution (3) result from the exact asymptotic behavior of the spectral density correlation function at the mobility edge,

$$R(\omega) \equiv \frac{1}{\nu_0^2}\langle \nu(\varepsilon)\nu(\varepsilon')\rangle - 1 = -c_d\beta^{-1}|x-x'|^{-2+\gamma}, \quad x\equiv\varepsilon/\Delta, \quad |x-x'|\gg 1, \tag{5}$$

where $\nu(\varepsilon)$ is the exact density of states at the energy $\varepsilon$, $\nu_0$ is the mean density of states, and $c_d$ is a positive number which depends only on the dimensionality $d>2$. The asymptotic relation (5) was obtained in Ref. 7 by calculating all the diagrams (with an accuracy to a numerical coefficient). This calculation could be carried out after taking into account the analytic properties of the diffusion propagator and certain scaling relations at the mobility edge.

To derive the result (3), we will use the effective "plasma model," as suggested by Dyson.[8] In such a model the level distribution is mapped onto the Gibbs distribution of a classical one-dimensional gas of fictitious "particles" with a repulsive pairwise interaction $f(|\varepsilon_i-\varepsilon_j|)$ in the presence of a confining potential $V(\varepsilon)$:

$$\mathscr{P}(\{\varepsilon_n\})=Z^{-1}\exp[-\beta\mathscr{H}(\{\varepsilon_n\})], \tag{6}$$

$$\mathscr{H}(\{\varepsilon_n\})=\sum_{i<j} f(|\varepsilon_i-\varepsilon_j|)+\sum_i V(\varepsilon_i). \tag{7}$$

Here $Z$ is the partition function, and $\beta$ plays the role of the inverse temperature. For $f(|\varepsilon_i-\varepsilon_j|)\equiv\ln|\varepsilon_i-\varepsilon_j|^{-1}$, Eqs. (6) and (7) reproduce exactly the level distribution in the RMT, where $\beta$ depends on the symmetry class, as described by Eq. (1). The choice $V(\varepsilon_i)=\varepsilon_i^2$ of the confinement potential leads to the Gaussian ensembles, but other choices are also possible.[4,9] In the metallic phase such a description is exact for the energy separation $|\varepsilon_i-\varepsilon_j|<E_c\equiv\hbar/\tau_D$, where $\tau_D=L^2/D$ is the "ergodic" time it takes an electron to diffuse through the system. At $t<\tau_D$, i.e., $|\varepsilon_i-\varepsilon_j|>E_c$, the level statistics are completely different[3] from those of the RMT. However, they are also described by the Gibbs distribution (6), (7), even though the pairwise interaction $f$ has a power-law asymptotic behavior.[10]

At the mobility edge $E_c\sim\Delta$ ($g=E_c/\Delta$ is a dimensionless conductance). Therefore, the energy separation of a few $\Delta$ is already outside the RMT region of validity. We will show that the asymptotic behavior of the correlation function (5) at the mobility edge is described correctly by the Gibbs model with the power-law interaction

$$f(|x-x'|)=\frac{1-\gamma}{2\pi c_d}\cot(\pi\gamma/2)|x-x'|^{-\gamma},\quad 0<\gamma<1,\quad x\equiv\varepsilon/\Delta. \tag{8}$$

(Naturally, this interaction is different from that in Ref. 10, where $\gamma>1$, which describes the *nonuniversal* level statistics in the metallic phase at the scale $\omega\gg E_c$.) Before proving this assertion, we will show how the form of the pairwise interaction governs the asymptotic behavior of $P(s)$.

The distribution $P(s)$ describes the probability of finding the nearest adjacent level at the distance $s=\omega/\Delta$ from a given level. It is equivalent to the probability of finding a "gap" of width $s$ (i.e., a region which contains no "particles") in the Gibbs model. This probability is obtained[4] from Eq. (6):

$$P(s)=\exp[-\beta(F_s-F_0)], \tag{9}$$

where $F_s$ is the free energy of the one-dimensional gas (7) which is distributed along the straight line with a gap $s$ near its center. For $s\gg1$, one introduces a continuous density $\rho_s(x)$ to describe such a distribution. In the mean-field approximation (MFA) $F_s$ can then be expressed as the functional

$$o_s(x)\rho_s(x')f(|x-x'|)+\int_{|x|>\frac{s}{2}} dx\,\rho_s(x)V(x), \tag{10}$$

where $\rho_s(x)$ obeys the mean-field (MF) equation

$$\int_{|x'|\geqslant\frac{s}{2}} dx'\,\rho_s(x')f(|x-x'|)=-V(x)-\mu_s,\quad |x|\geqslant s/2. \tag{11}$$

Here $\mu_s$ arises from the "particle" number conservation (which corresponds to the level number conservation in the original quantum disordered system),

$$\int_{|x|\geqslant\frac{s}{2}} dx\,\rho_s(x)dx=\int_{-\infty}^{\infty}\rho_0(x)=\mathcal{N}. \tag{12}$$

Setting $s=0$ in Eqs. (10) and (11), we find the density $\rho_0(x)$ and the free energy $F_0$ for a uniform distribution.

Equations such as (10) and (11) were derived for a circular ensemble (a classical gas with the $\log|x-x'|^{-1}$ interaction confined to a circular wire) by Dyson[8] (see also Ref. 4), who has also found that corrections to the MF solution [with allowance for the entropy term added to the functional (10) and for the discreteness of the original distribution] lead to a linear in $s$ contribution to the difference $F_s-F_0$. For $s\gg 1$ this contribution is small compared to the leading quadratic term, Eq. (4). We will consider only the leading terms in the $s\gg 1$ limit, which are described by the MFA.

For the circular ensemble[8] there is no need for the confining potential $V(x)$. We use the linear ensemble which is more convenient for relating the interaction $f(|\varepsilon_i-\varepsilon_j|)$ to the correlation function (5). Equation (11) with the weakly singular kernel (8) can be solved for any $V(x)$. This exact solution shows that $\rho_s$ and $\rho_0$ (and here $F_s$ and $F_0$) depend strongly on $V(x)$ [see Eq. (17) for $\rho_0$]. It is easy to show, however, that the difference $F_s-F_0$, and hence the distribution (9), do not depend on $V$ for $s\gg 1$ in the limit $\mathcal{N}\to\infty$. Furthermore, the asymptotic relation for this universal distribution can be found, within a numerical coefficient, without knowing the explicit solution of Eq. (11).

The explicit dependence of $F_s-F_0$ on $V(x)$ [Eq. (10)] can be eliminated with the help of Eqs. (11) and (12). After some transformations which take into account that the "chemical potential" changes due to the gap formation $\mu_s-\mu_0\sim s/\mathcal{N}\ll 1$, we find, with accuracy up to $s/\mathcal{N}$, that

$$F_s-F_0=-\frac{1}{2}\int_{|x|\geqslant\frac{s}{2}} dx\int_{|x'|\geqslant\frac{s}{2}} dx'\delta\rho_x\delta\rho(x')f(|x-x'|)$$
$$+\frac{1}{2}\int_{-s/2}^{s/2} dx\int_{-s/2}^{s/2} dx'\rho_0(x)\rho_0(x')f(|x-x'|), \tag{13}$$

where $\delta\rho(x)=\rho_s(x)-\rho_0(x)$ decreases rapidly for $x\gg s$. The function $\delta\rho(x)$ obeys the MF equation:

$$\int_{|x'|\geqslant\frac{s}{2}} dx'\delta\rho(x')f(|x-x'|)=\int_{-s/2}^{s/2} dx'\rho_0(x')f(|x-x'|),\quad |x|\gg s/2, \tag{14}$$

which follows from Eq. (11), if the small term $\mu_s-\mu_0\sim s/\mathcal{N}$ is disregarded. The uniform level density $\rho_0(x)$ in Eqs. (13) and (14) still depends on $V(x)$, Eq. (11). However, for $|x|<s/2\ll\mathcal{N}$, this dependence is negligible, and in the limit $\mathcal{N}\to\infty$ we

find $\rho_0 = 1$ (in units of $1/\Delta$). We clearly see from Eqs. (13) and (14) that the quantity $F_s - F_0 = -\beta^{-1} \ln P(s)$ is determined exclusively by the interaction $f(|x-x'|)$.

Equations (13) and (14) are valid for an arbitrary long-range interaction $f(|x-x'|)$. For the power-law interaction (8) (with $0<\gamma<1$) we can rescale $x \to sx$ and $x' \to sx'$ to find that the solution of Eq. (14) (with $\rho_0 = \text{const}$) has the form $\delta\rho(x) = \varphi(x/s)$, where $\varphi(x)$ is a universal function which does not depend on $s$. Substituting this solution in Eq. (13), we obtain the following expression after the same rescaling:

$$F_s - F_0 = -\beta^{-1} \ln P(s) = A_\gamma s^{2-\gamma}, \tag{15}$$

where the universal constant $A_\gamma$ depends only on the power of the interaction. Calculating this constant for the limiting case ($\gamma=0$) of the logarithmic interaction in Eqs. (13) and (14), we reproduce the asymptotic relation (4) known from the RMT,[4] which leads to the asymptotic relation (3).

To prove that the interaction (8) reproduces the correlation function (5), we use the relation[9]

$$R(x,x') = -\beta^{-1} \frac{\delta\rho_0(x)}{\delta V(x')}. \tag{16}$$

For the interaction $f = a_\gamma |x-x'|^{-\gamma}$ and an arbitrary $v(x) \equiv V(x) + \mu_0$ the solution of Eq. (11) with $s=0$ is found, using the methods described in Ref. 11, as follows:

$$\rho_0(x) = \frac{\cos^2(\pi\gamma/2)(x+D)^{(\gamma-1)/2}}{\pi^2 a_\gamma} B\left(\gamma, \frac{1-\gamma}{2}\right) \frac{d}{dx} \left\{ \int_x^D dt (t+D)^{1-\gamma} (t-x)^{(\gamma-1)/2} \right.$$
$$\left. \times \frac{d}{dt} \int_{-D}^{t} d\tau (\tau+D)^{\gamma-1/2} (t-\tau)^{(\gamma-1)/2} v(\tau) \right\}. \tag{17}$$

Here $B$ is the Euler function, and $D$ is the band edge which is found from Eq. (12) which tends to infinity when $\mathcal{N} \to \infty$. Using the variational derivative (16), i.e., substituting $-\beta^{-1}\delta(\tau - x')$ for $v(\tau)$ in Eq. (17), we find the following relation in the limit $D \to \infty$:

$$R(x,x') = -\beta^{-1} \frac{1-\gamma}{2\pi a_\gamma} \cot\left(\frac{\pi\gamma}{2}\right) |x-x'|^{\gamma-2}. \tag{18}$$

Thus the Gibbs model with the power-law interaction gives the asymptotic behavior (5) of the correlation function. Comparing Eqs. (5) and (18), we obtain Eq. (8).

Note that for all three universal statistics in the metal phase, in the insulating phase and at the mobility edge a simple relation between the variance of the level number fluctuations in the limit (2) and the asymptotic behavior of the nearest-level-spacing distribution holds. Specifically, if the variance is proportional to $\langle N \rangle^\gamma$, then $-\ln P(s) \propto s^{2-\gamma}$. The linear in $\langle N \rangle$ variance is forbidden at the mobility edge[7] by the exact sum rule, which is due to the conservation of the total number of states $\mathcal{N}$. Therefore, the Poisson (i.e., the linear in $s$) asymptotic behavior of $P(s)$ is equally forbidden at the mobility edge. Finally, following Ref. 6, we note that for $s \ll 1$ the distribution $P(s)$ shows at the mobility edge the same behavior as that in the metallic phase, $P(s) \sim s^{\beta}$, which follows from the general symmetry theorem proved by Dyson.[8] The total distribution can then be described by the relation

$$P(s) = Bs^{\beta} \exp(-A_d \beta s^{2-\gamma}), \tag{19}$$

where $B$ is found from the normalization conditions.


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