# Exact solution of the Blume–Emery–Griffiths model on a Bethe lattice

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The Blume–Emery–Griffiths model is analyzed in the subspace of exchange-interaction constants $K=-\ln\cos hJ$. Exact expressions are derived for the free energy, for the $\lambda$-line of second-order phase transitions, for the tricritical point, and for the critical exponents.

In a study of the spin-1 Ising model proposed by Blume, Emery, and Griffiths[1] on hexagonal and Kagomé lattices, Horiguchi showed that when the constants of the dipole ($J$) and quadrupole ($K$) exchange interactions satisfy the condition

$$K=-\ln\cos hJ, \tag{1}$$

the model reduces to a spin-1/2 Ising model, but the single critical point for a second-order phase transition is replaced by an entire $\lambda$-line.[2–4] The same result, under condition (1), has been derived on a square lattice.[5]

In the present letter we examine a ferromagnetic Blume–Emery–Griffiths model on a Bethe lattice with an infinite-dimensional Hausdorff dimensionality.[6–10] We examine the critical properties of the model under the Horiguchi condition (1). We find an exact analytic expression for the free energy. In this model we find a $\lambda$-line of second-order phase transitions, which terminates in a tricritical point at lattice coordination numbers $q\geqslant 6$ (Fig. 1).

The partition function of the Blume–Emery–Griffiths model is

$$Z=\sum_{\{S\}}\exp\left[\sum_{\langle ij\rangle}(JS_iS_j+KS_i^2S_j^2)+\sum_i(hS_i-\Delta S_i^2)\right], \tag{2}$$

where $S_i=0,\ \pm 1$; the first summation within the exponential function is over all lines of the lattice; the second is over all sites; and the summation outside the exponential function is over all configurations of the system.

On a Bethe lattice, partition function (2) can be written

$$Z_n=\sum_{S_0}\exp[hS_0-\Delta S_0^2]\,[g_n(S_0)]^q, \tag{3}$$

where $q$ is the coordination number of the lattice, $S_0$ is the spin at the central site, $g_n(S_0)$ is the partition function on an individual branch of the Bethe lattice, and $n$ is the number of shells of the lattice.[11]

For $g_n(S_0)$ we have the recurrence relation

  

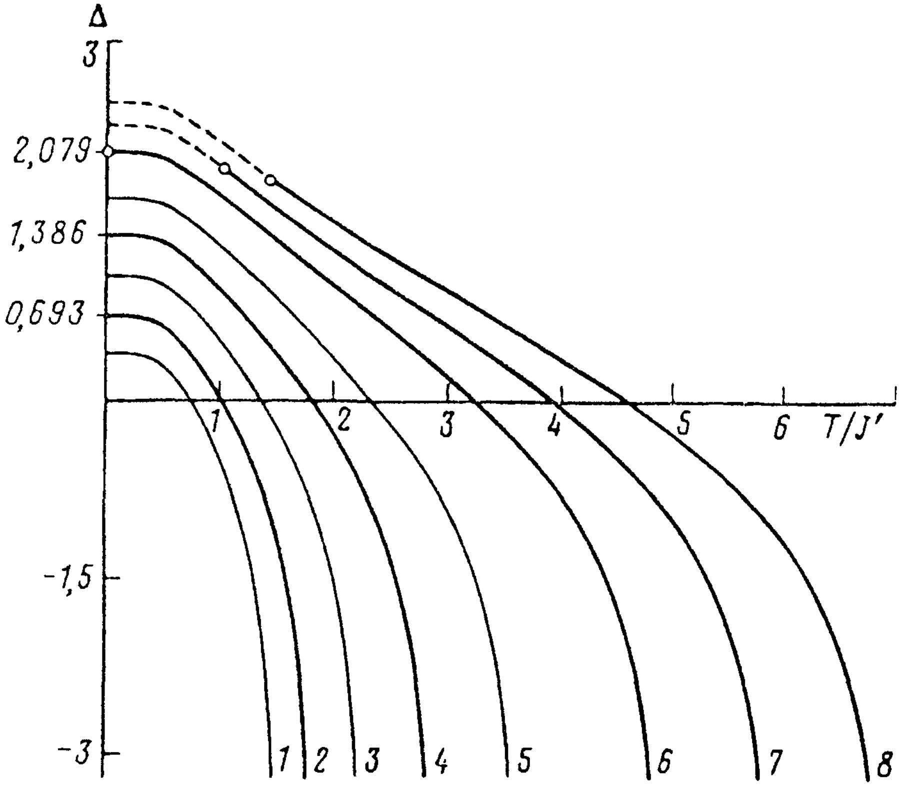


FIG. 1. λ-lines of second-order phase transitions for various lattices. 2, 4, 6, 7, 8—Bethe lattice with coordination numbers $q$ of 3, 4, 6, 7, and 8, respectively; 1, 3, 5—hexagonal, square, and triangular lattices. The circles on curves 6, 7, and 8 are tricritical points.

$$g_n(S_0)=\sum_{S_1}\exp[JS_1S_0+KS_1^2S_0^2+hS_1-\Delta S_1^2]\,[g_{n-1}(S_1)]^{q-1}\,, \tag{4}$$

where $S_1$ is the value of the spin at the site nearest $S_0$.

To describe this model we can introduce two order parameters: the magnetization $[m=(1/N)\Sigma_{i=1}^N\langle S_i\rangle]$ and the quadrupole moment $[p=(1/N)\Sigma_{i=1}^N\langle S_i^2\rangle]$, which are coupled with the external fields $h$ and $\Delta$ . Here $N$ is the number of lattice sites.

In the case in which the solutions of recurrence relations (4) converge on a stable point[11] in the thermodynamic limit ($n \to \infty$), the thermodynamic properties of the model are governed completely by the equations

$$\exp(2h)=\left(\frac{1+v}{1-v}\right)^{\gamma+1}\frac{v^2(1-p)+v-p\tanh J}{v^2(1-p)-v-p\tanh J}, \tag{5a}$$

$$\exp(2\Delta)=4(1-v^2)^{\gamma+1}\frac{(1-p)^2}{p^2-m^2}, \tag{5b}$$

$$(1-p)v^3-mv^2-(1+p\tanh J)v-m\tanh J=0, \tag{5c}$$

where

$$v\equiv\frac{g(-1)-g(+1)}{2g(0)},\quad \gamma=q-1.$$

The free energy of the model per site is

$$\frac{f}{k_BT}=\frac{\gamma-1}{2}\ln(1+v^2\coth J)$$
$$-\ln\left|\frac{1-\exp(2h)[(1+v)/(1-v)]^\gamma}{1-v\coth hJ-(1+v\coth J)\exp(2h)[(1+v)/(1-v)]^\gamma}\right|. \tag{6}$$

At a zero field $h$—this case corresponds to extrema of the Gibbs free energy—Eqs. (5) have two solutions:

$$v=0,\quad \exp(\Delta)=2\frac{1-p}{p},\quad m=0, \tag{7a}$$

and

$$1-\gamma\tanh J\cdot p+\frac{\gamma}{6}[3(\gamma+1)-6p-(\gamma-1)(\gamma-2)\tanh J\cdot p]v^2+O(v^4)=0. \tag{7b}$$

At the point of the second-order phase transition, the extrema of the Gibbs free energy coincide. From the intersection of solutions (7a) and (7b) we therefore find

$$\Delta_\lambda=\ln[2(\gamma\tanh J-1)], \tag{8a}$$

$$p_\lambda=\frac{1}{\gamma\tanh J}. \tag{8b}$$

These expressions govern the $\lambda$-line of second-order phase transitions. The condition

$$\left.\frac{\delta\Delta}{\delta p}\right|_{\substack{m=0\\v=0}}=\left.\frac{\partial\Delta}{\partial p}\right|_{\substack{m=0\\v=0}}+\left.\frac{\partial\Delta}{\partial v^2}\frac{\partial v^2}{\partial p}\right|_{\substack{m=0\\v=0}}+\left.\frac{\partial\Delta}{\partial m}\frac{\partial m}{\partial p}\right|_{\substack{m=0\\v=0}}=0$$

on the second solution, (7b), cuts off the $\lambda$-line at the tricritical point, which has the value

$$\tanh J_{Tr} = \frac{3}{\gamma-2}, \tag{9a}$$

$$p_{Tr} = \frac{\gamma-2}{3\gamma}. \tag{9b}$$

It can be seen from expression (9a) that in the ferromagnetic Blume–Emery–Griffiths model a tricritical point exists if $q \geqslant 6$.

The expansions for the fields $h$ and $\Delta$ near the $\lambda$-line take the following form when we use (8):

$$h = m\left\{ -\frac{\gamma}{\gamma+1}\frac{1}{p_\lambda^2}(p-p_\lambda) + \frac{\gamma}{3(\gamma+1)^3}\frac{1}{p_\lambda^3}[(\gamma+2)(\gamma+1)-3(\gamma p_\lambda+1)]m^2 \right.$$
$$\left. + O[(p-p_\lambda)^2,\ m^2(p-p_\lambda),\ m^4] \right\}, \tag{10a}$$

$$\Delta-\Delta_\lambda = -\frac{1}{p_\lambda(1-p_\lambda)}(p-p_\lambda) + \frac{\gamma}{\gamma+1}\frac{1}{2p_\lambda^2}m^2 + O[(p-p_\lambda)^2, m^2(p-p_\lambda), m^4]. \tag{10b}$$

Since on the $\lambda$-line ($h=0$, $\Delta=\Delta_\lambda$) we have $(p-p_\lambda)\sim m^2$, expansions (10) can be rewritten as follows:

$$h = h_1 m^3 + h_2 m^5 + O(m^7), \tag{11a}$$

$$\Delta-\Delta_\lambda = \Delta_1 m^2 + \Delta_2 m^4 + O(m^6), \tag{11b}$$

where

$$h_1 = \frac{\gamma(\gamma-1)}{6(\gamma+1)^3}\frac{1}{p_\lambda^3}[3\gamma p_\lambda-\gamma+2], \quad \Delta_1 = -\frac{(\gamma-1)}{6(\gamma+1)^2}\frac{[3\gamma p_\lambda-\gamma+2]}{p_\lambda^2(1-p_\lambda)}.$$

At the tricritical point we have $h_1=0$ and $\Delta_1=0$ and thus $m\sim|h|^{1/5}$ ($\delta_{Tr}=5$), $m\sim|\Delta-\Delta_{Tr}|^{1/4}$ ($\beta_{Tr}=1/4$), $m\sim|T-T_{Tr}|^{1/2}$ ($\beta=1/2$), $p-p_{Tr}\sim|h|^{2/5}$, $p-p_{Tr}\sim|\Delta-\Delta_{Tr}|^{\beta_{2Tr}}$ ($\beta_{2Tr}=1/2$), and $p-p_{Tr}\sim|T-T_{Tr}|^{\beta_2}$ ($\beta_2=1$).

Interestingly, under the Horiguchi condition the ferromagnetic Blume–Emery–Griffiths model contains a tricritical point at $q \geqslant 6$, while a tricritical point has not been observed on 2D lattices.[2-5]

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