Instability and self-focusing of solitons in the boundary layer

A. N. D'yachenko and E. A. Kuznetsov
L. D. Landau Institute for Theoretical Physics, Moscow 117334, Russia

(Submitted 16 December 1993)
Pis'ma Zh. Eksp. Teor. Fiz. **59**, No. 2, 103–108 (25 January 1994)

It is demonstrated both analytically and numerically that in the limit of small viscosity the instability of one-dimensional long-wave solitons in the boundary layer results in their self-focusing and collapse. Theoretical predictions are in a qualitative agreement with the experiments.[1]

1. In the nonlinear acoustics there exists an analog of the self-focusing of light. It involves the collapse of acoustic waves with positive dispersion[2] which can be considered as the nonlinear stage of the Kadomtsev–Petviashvili (KP) instability of one-dimensional solitons.[3] In contrast with light self-focusing, the mechanism of the KP instability (and hence of the collapse) is connected with a decrease in the soliton velocity with increasing pulse amplitude.

In this letter we show that one-dimensional solitons which propagate in the boundary layer and which represent holes in the mean velocity profile undergo an instability of the winding type analogous to that of acoustic solitons. This instability is of the focusing type which leads at the nonlinear stage to the separation of a 1D soliton into individual clusters and to the ensuing self-focusing of each cluster. We show that both theoretical and numerical results presented in this paper are in a qualitative agreement with the experimental data on the coherent structures in the boundary layer.[1]

2. We consider a shear flow with a constant velocity U which is parallel to the plane of the wall and which depends on the normal coordinate z. We assume that 1) the function U has no inflection points, 2) U tends to the constant U_0 as $z\to\infty$, and 3) the first derivative U' at $z=0$ is positive, $U'(0)>0$. The given velocity profile $U(z)$ is stationary only for an ideal fluid. For a sufficiently small viscosity it is not stationary: The boundary layer becomes turbulent, but the mean value of the velocity $\overline{U(z)}$ behaves in the same way. In this case, however, the boundary layer thickness h depends on the distance x from the edge of the blowing plate. This thickness is given by the expression (see Ref. 4)

$$h\propto\frac{x}{\log \mathrm{Re}},$$

where $\mathrm{Re}=U_0\, h/\nu$ is the Reynolds number. Let us consider the low-frequency oscillations (the wavelength $\lambda\gg h$), which are insensitive to the high-frequency turbulent fluctuations. We ignore the dependence of h on x which holds for large values of $\log \mathrm{Re}\gg 1$ $(x\gg h)$. In a weak, nonlinear approximation the equation for the amplitude $A(x, y, t)$ in this case can be derived from the Euler equation:

$$\frac{\partial A}{\partial t}+U'(0)AA_x+U_0\,h\,\frac{\partial}{\partial x}\,\hat{k}A=0, \qquad (1)$$

where $h=U_0/U'(0)$. The Fourier transform of the integral operator $\hat{\mathrm{k}}$ is the modulus $|\mathrm{k}|=(k_x^2+k_y^2)^{1/2}$. The amplitude A in the leading order is connected with the velocity fluctuations along the mean flow (parallel to x) by means of $v_x\approx A(x,\,y,\,t)U'(z)$. Equation (1) was derived first by Shrira.[5] It represents a two-dimensional generalization of the well-known Benjamin–Ono (BO) equation which describes the long waves in the stratified liquids. It should be noted that for this problem this equation in the 1D case was derived first in Ref. 6, taking into account the small viscosity. Equation (1) is written with an accuracy of $(kh)^3$. In the next order the imaginary contribution persists to a frequency $w=U_0\,hk_x|k|$ providing a weak damping.

3. By the usual rescaling Eq. (1) can be written in the standard form of the BO equation:

$$u_t=\frac{\partial}{\partial x}\,\hat{k}u-6uu_x=\frac{\partial}{\partial x}\,\frac{\delta H}{\delta u}, \qquad (2)$$

where the Hamiltonian is

$$H=\frac{1}{2}\,I_1-I_2\quad\left(I_1=\int u\hat{k}u\,d\mathbf{r},\quad I_2=\int u^3 d\mathbf{r}\right).$$

Soliton solutions of (2) traveling along x, $u=u_s(x-Vt,\,y)$ are stationary points of H for a fixed x projection of the momentum $P_x=1/2\int u^2 d\mathbf{r}$,

$$\delta(H-VP_x)=0\quad\text{or}\quad -Vu_s-\hat{k}u_s+3u_s^2=0. \qquad (3)$$

For the 1D case the solution of Eq. (3) can be found explicitly:

$$u_s=2V/3(x^2V^2+1)^{-1}\quad(V>0). \qquad (4)$$

In the 2D case Eq. (3) has a broader class of solutions. We are interested only in the ground soliton as a solution of (3) without nodes (it was found numerically in Ref. 7). According to the Lyapunov theorem, solitons are stable if they realize the minimum of H. Under scaling transformations, the remaining $P_x, u_s\,(\mathbf{r})\to(1/a)^{d/2}u_s(\mathbf{r}/a)$, H becomes a function of the scaling parameter a:

$$H(a)=\frac{I_{1s}}{2a}-\frac{I_{2s}}{a^{d/2}} \qquad (5)$$

(here and below the subscript s denotes values of the integrals on the soliton solution). We see, therefore, that in the 1D case $H(a)$ has a minimum which corresponds to the soliton. To show that 1D soliton realizes the precise minimum of H against all perturbations, we will use the inequality[8]

$$\int u^3 d\mathbf{r}\leqslant C\left(\int u\hat{k}u\,d\mathbf{r}\right)^{d/2}\left(\int u^2 d\mathbf{r}\right)^{\frac{3-d}{2}}, \qquad (6)$$

where the constant $C=I_{2s}I_{2s}^{-d/2}(2P_{xs})^{(d-3)/2}$. Substituting estimate (6) into H and setting $P_x=P_{xs}$, we obtain the following inequality for H in the 1D case:

$$H \leqslant H_s + 1/2(I_1^{1/2} - I_{1s}^{1/2})^2, \quad (7)$$

which becomes precise on the 1D soliton.

4. For both the 1D and 2D cases the solitons move in the upstream direction which is opposite to that in which the linear waves propagate: $v_{gr}V<0$. They represent holes in the mean velocity profile and therefore move slowly as they increase their amplitude. Thus, for a soliton weakly modulated in the transverse direction the regions with a small amplitude overtake those with a large amplitude. This process obviously leads to a self-focusing instability of the soliton front. This instability is analogous to the Kadomtsev–Petviashvili instability of the acoustic solitons.[3] The growth rate for the instability can be easily determined in the long-wave limit. Omitting all calculations (they are similar to those in Ref. 9), we present the final result for the growth rate:

$$\gamma^2 = \frac{k_y^2 V^2}{2} > 0, \quad (8)$$

where k_y is the wave number of the perturbations. The development of an instability gives rise to a soliton separation into individual clusters with a typical size on the order of that for the soliton. There exist several possibilities for further evolution of each cluster. First of all, the formation of a 2D soliton from such a cluster is impossible, because the Hamiltonian on the 2D soliton is identically equal to zero (this follows from $\partial_a H|_{a=1}=0$). Secondly, by the same reason, clusters cannot disappear due to the dispersion: The initial state in the form of 1D soliton (plus small perturbations) has a negative H, but the broadening distribution has a positive H. At the same time, as follows from (5), for states with a negative H the Hamiltonian under scaling transformations in the 2D case becomes unbounded from below as $a \to 0$. One should therefore expect a collapse, which is similar to the falling down of a particle in an unbounded self-consistent potential. There is no alternative to the collapse in the case under consideration. It is confirmed, in particular, by the following estimate, which is correct for an arbitrary region Ω with a negative Hamiltonian H_Ω (compare with Refs. 10 and 11:

$$\max_{x\in\Omega} u \geqslant \frac{|H_\Omega|}{2P_{x\Omega}}. \quad (9)$$

We can see, therefore, that max u, plotted as a function of t, is always bounded by a conservative value ($H<0$!). The vanishing of the initial maximum, or even its decrease is therefore impossible. If initially in some separate region $H_\Omega<0$, and if this region emits radiation, then because the radiative waves account for the positive portions of both H and P_x, the Hamiltonian of this region becomes progressively more negative and increases its absolute value, but $P_{x\Omega}$ as a positive value decreases. In accordance with (9), this situation leads to an increase of max u. The radiation therefore promotes collapse.

Equation (2) in the 2D case and the two-dimensional nonlinear Schrödinger equation (NLSE) manifest the same critical behavior under the scaling transformations; i.e., the Hamiltonians for both systems under these transformations behave as homogeneous functions of the scaling parameter a. Our system should therefore be called critical similar to the 2D NLSE. Note also that for the critical NLSE (see, for example, Ref. 13) the energy (or the number of particles) contributing to the singularity during the collapse is finite and the shape of the collapsing cluster coincides with the shape of the 2D soliton. By analogy, one should expect the same critical behavior for the system under consideration. It is easy to see that Eq. (2) can be satisfied by the self-similar collapsing solution (more exactly—substitution) of the form $u=(t_0-t)^{-1/2}f[\mathbf{r}(t_0-t)^{-1/2}]$. On this solution the energy, which coincides with P_x within a constant multiplier, does not depend on t (if the integral converges).

As for the critical NLSE (see, for example, Refs. 12 and 13), it is possible to introduce the notion of the critical power $P_{x,\mathrm{cr}} = P_{xs}$ which is given by the ground soliton and which is independent of V.

Using inequality (6) at $d=2$, we obtain the following estimate for H:

$$H \geqslant \frac{I_1}{2}\left[1-\left(\frac{P_x}{P_{x,cr}}\right)^{1/2}\right]. \tag{10}$$

This means that the Hamiltonian for fixed $P_x < P_{x,\mathrm{cr}}$ has a lower bound and reaches its lower boundary on the solutions with mean $\langle k \rangle$ tending to zero. For initial conditions with $P_x \leqslant P_{x,\mathrm{cr}}$, the long-time asymptotic state manifests the distribution broadening due to the dispersion.

5. To check the proposed theory, we carried out a straightforward numerical integration of Eq. (2). The equation was solved with the help of FFT in the domain $\{-4\pi < x < 4\pi\,;\, 0 < y < 4\pi\}$, which is symmetrical with respect to y ($y \to -y$), with the boundary conditions:

$$u_{|x=-4\pi} = u_{|y=4\pi} = 0; \quad \hat{k}u_{|x=4\pi} = 0.$$

Such boundary conditions do not conserve the x momentum on the Hamiltonian and permit small-amplitude waves to leave the counting region. The initial conditions were chosen in the Lorentzian form with two varying widths, $1/V_x$ and $1/V_y$, along x and y, respectively,

$$u(\mathbf{r}) = 2/3\,\frac{|\mathbf{V}|}{(\mathbf{Vr})^2+1}\,.$$

The variation of $\mathbf{V}$ allows us to change the initial values of H from positive to negative values. For all initial conditions with negative H we observed the collapse. As one can see from Fig. 1, the amplitude of the peak moving with acceleration increases. The temporal behavior of the peak velocity and of the peak amplitude are familiar, which indicates that the pulse collapse is of a self-similar nature. Upon approaching the singularity the peak anisotropy vanishes, and the peak distribution becomes nearly symmetric (Fig. 2). The x momentum remains apaproximately constant in each run with a negative H. At the same time, the Hamiltonian decreases (Fig. 3). For the initial conditions with $P_x < P_{x,\mathrm{cr}}$ ($H > 0$) we observe a slow evolution: The distribution of u near the maximum has the shape similar to the two-dimensional soliton and it slowly decays.

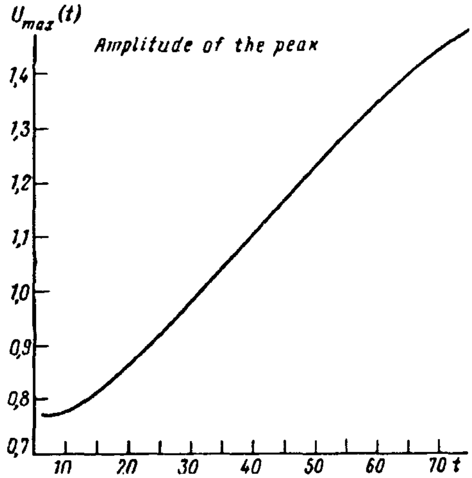

FIG. 1.

6. In conclusion, we would like to point out several interesting experiments published in Ref. 1, which summarize the results of many years of experimental studies of the onset of coherent structures in the boundary layer of the blowing plate by the mechanical vibrating system near the edge of the plate (see also Ref. 14). According to these experimental data, the one-dimensional solitons are excited in the initial stage, later (for larger distances from the plate edge) one-dimensional solitons demonstrate the instability which results "in the formation of thorns," i.e., localized three-dimensional coherent structures. A self-focusing of the above structures is observed at longer distances. A later stage of the development of thorns-solitons leads to the formation of vortices and to their eventual separation.

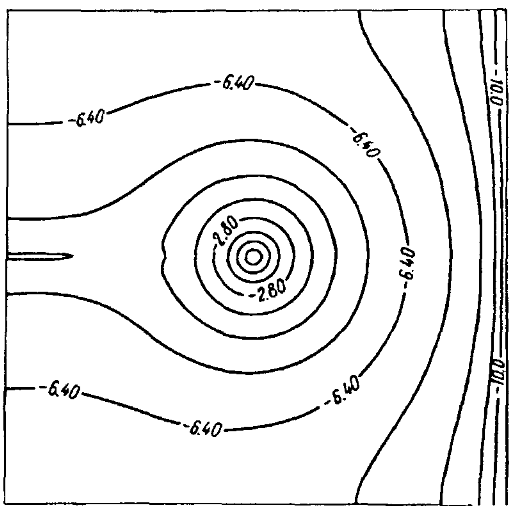

FIG. 2.

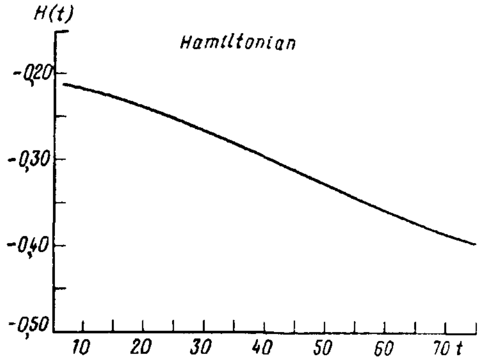

FIG. 3.

The above theory and the numerical experiments explain all these experimental observations, but not the formation of vortices, for which Eq. (1) is inapplicable. Until now it has not been clear whether it is capable of describing the given experiment quantitatively. Here we should note that comparison of the theoretical results based on the analysis of the one-dimensional model (1), i.e., in the framework of 1D BO equation,[6] showed rather good correspondence with this experiment. It is hoped, therefore, that a quantitative agreement between the 3D theory presented here and the experiment can be reached.

We wish to thank V. Shrira and Yu. Kachanov for useful discussions. This work was supported, in part, by the Russian Foundation for Basic Research and the Landau–Weizmann Program. K. E. thanks the Weizmann Institute of Science for its hospitality.

[1] Yu. S. Kachanov and O. S. Ryzhov, Sib. Fiz.-Tekh. Journ. N1, 34–52 (1992).
[2] E. A. Kuznetsov, S. L. Musher, and A. V. Shafarenko, JETP Lett. **37**, 241 (1983).
[3] B. B. Kadomtsev and V. I. Petviashvili, Sov. Phys. Dokl. **15**, 539 (1970); B. B. Kadomtsev, *Collective Phenomena in Plasma*, Nauka, Moscow (1976).
[4] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*, Pergamon Press, 1959.
[5] V. I. Shrira, Doklady AN SSSR **308**, 732 (1989).
[6] V. I. Zhuk and O. S. Ryzhov, Doklady AN SSSR, **263**, 56 (1982); O. S. Ryzhov. Zh. Vych. Mat. i Mat. Fiz. **29**, 1804 (1990).
[7] A. A. Abramyan, Ya. A. Stepanyants, and V. I. Shrira, Doklady AN SSSR **327**, 460 (1992).
[8] L. Neirenberg, Ann. Sci. Norm. Sup. Pisa. **20**, N 4, 733 (1966).
[9] E. A. Kuznetsov and S. K. Turitsyn, Sov. Phys. JETP **55**, 844 (1982).
[10] V. E. Zakharov, In *Handbook of Plasma Physics 3*, ed. A. Galeev, R. Sudan (Elsevier, NY, 1984).
[11] E. A. Kuznetsov and S. L. Musher, Sov. Phys. JETP **64**, 947 (1986).
[12] S. N. Vlasov, V. A. Petrishchev, and V. I. Talanov, Izv. VUZ. Radiofizika **12**, 1353 (1970) [Radiophys. Quantum Electron. **14**, 1062 (1974)].
[13] D. Dyachenko, A. C. Newell, A. Pushkarev, and V. E. Zakharov, Physica **57D**, 96 (1992).
[14] V. N. Borodulin and Yu. S. Kachanov, Izv. Sib. Branch USSR Ac. Sci. N18, 65 (1988).

Published in English in the original Russian journal. Reproduced here with the stylistic changes by the Translations Editor.