Nonlocal Bell's paradoxes for an arbitrary number of observers

A. V. Belinskiĭ
Physics Faculty, M. V. Lomonosov Moscow State University, 119899 Moscow, Russia

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Bell's inequalities are proved for 2 and N observers. The Greenberger–Horn–Zeilinger paradox is formulated without the assumption of locality. The formulation is based on solely the existence of a positive definite probability distribution function.

The violation of Bell's inequalities, which is predicted by quantum theory and which has been verified experimentally many times, is often interpreted as a manifestation of the nonlocal nature of quantum theory. The idea here is that Bell's inequalities in their original form[1] were derived from concepts of the *theory of hidden parameters*,[2] *one* assumption of which is *locality*, i.e., the absence of an effect of two spatially separated measurement instruments on each other. On the other hand, it has been shown in several places[3–7] that Bell's inequalities for two observers can be justified only if one allows the existence of a positive definite probability distribution. In this letter we propose a new algorithm for proving nonlocal versions of Bell's inequalities ("nonlocal" in the sense that they do not use the assumption of locality). This algorithm makes it possible to rigorously prove Bell's inequalities for an arbitrary number of observers N and also to formulate a *nonlocal Greenberger–Horn–Zeilinger paradox* of the $+1=-1$ type.

We consider a random process described by $2N$ dichotomic variables which take on unit values:

$$A_1=\pm 1,\quad A_1'=\pm 1,\quad A_2=\pm 1,\quad A_2'=\pm 1,\ldots,\quad A_N=\pm 1,\quad A_N'=\pm 1. \tag{1}$$

We assume that there exists a normalized positive definite probability distribution

$$W(A_1, A_1', A_2, A_2', \ldots, A_N, A_N')\geqslant 0, \tag{2}$$

$$\sum_{A_1}\sum_{A_1'}\ldots\sum_{A_N'}(A_1, A_1', A_2, A_2', \ldots, A_N, A_N')=1, \tag{3}$$

which satisfies correspondence conditions

$$\sum_{A_1} W(A_1, A_1', A_2, A_2', \ldots, A_N, A_N')=W(A_1', A_2, A_2', \ldots, A_N, A_N'), \tag{4}$$

with similar conditions for other variables and distributions of lower dimensionalities.

We begin by proving Bell's inequalities for two observers ($N=2$) in the form

$$|\Pi| \equiv (1/2)\,|\langle A_1 A_2\rangle + \langle A_1' A_2\rangle + \langle A_1 A_2'\rangle - \langle A_1' A_2'\rangle| \leqslant 1, \tag{5}$$

without the assumption of locality. In an experiment to test (5), the average would be taken over successive realizations.

The discrete probability distribution in (2) with $N=2$ consists of 2^4 joint probabilities:

$$W_1(++++) \equiv W(A_{11}=+1,\ A_{11}'=+1,\ A_{12}=+1,\ A_{12}'=+1),$$

$$W_2(+++-) \equiv W(A_{21}=+1,\ A_{21}'=+1,\ A_{22}=+1,\ A_{22}'=-1),$$

etc., up to $W_{16}(----)$. We have thus enumerated the joint probabilities $W_M(A_{M1}, A_{M1}', A_{M2}, A_{M2}')$, $M=1, 2, ..., 16$. Normalization condition (3) now becomes

$$\sum_M W_M = 1. \tag{6}$$

We write the average product of two variables, e.g., A_1 and A_2, as a sum,

$$\langle A_1 A_2\rangle = \sum_M A_{M1} A_{M2} W_M, \tag{7}$$

with corresponding expressions for the other averages in (5). We thus have

$$\Pi \equiv (1/2)(\langle A_1 A_2\rangle + \langle A_1' A_2\rangle + \langle A_1 A_2'\rangle - \langle A_1' A_2'\rangle) = \sum_M S_{M2} W_M, \tag{8}$$

where a Bell's observable for two observers,

$$S_{M2} \equiv (1/2)[(A_{M2}+A_{M2}')A_{M1} + (A_{M2}-A_{M2'})A_{M1}'], \tag{9}$$

can take on only unit values, ± 1, by virtue of (1). Expression (5) thus follows from (2), (6), and (8); half (eight) of the terms in sum (8) appear with a plus sign, and half with a minus sign [if all terms had the same sign, this sum would be simply an expansion of unity, (6)].

This proof is actually a generalization of the derivation of a local Bell's inequality[8–10] to the nonlocal case with a positive definite probability distribution. A natural further generalization is to go over from two observers to an arbitrary number of observers N. In this case the normalized positive definite probability distribution in (2)–(4) has 2^{2N} components W_M, $M=1, 2, ..., 2^{2N}$.

We introduce a Bell's observable of the type

$$S_{MN} = (1/2)[(A_{MN} \pm A_{MN}')S_{MN-1} \pm (A_{MN} \mp A_{MN}')S_{MN-1}'] = \pm 1, \tag{10}$$

which is analogous to that used in Refs. 9 and 10 to derive local Bell's inequalities. Recurrence relation (10) makes it possible to go over from Bell's inequalities for $N=2$ to those for $N=3$, etc. The prime in the last term in (10) means that the "primed" variables which appear in a Bell's observable for $N-1$ observers have been changed to "unprimed" variables, and vice versa. According to (9), we have $S_{M1}=A_{M1}$. The signs in (10) are arbitrary, but if there is a sum in the first set of parentheses, then there must be a difference in the second set, and vice versa.

From (2), (6), and (10) we have

$$|\sum_{M} S_{MN}W_M|\leqslant 1. \tag{11}$$

This is (we wish to stress) a prototype nonlocal Bell's inequality for an arbitrary number of observers N. An explicit expression for it is calculated from (10). Here we must carry out the following formal transformations: eliminate the indices M from (10), express S_{N-1} and S'_{N-1} in terms of variables (1), expand all the parenthetical expressions, and include each term in an averaging operation. The resulting expression must not be greater than one in absolute value. For example, for $N=3$, one version of the combination of signs in (10) leads to

$$(1/2)|(\langle A_1' A_2 A_3\rangle+\langle A_1 A_2' A_3\rangle+\langle A_1 A_2 A_3'\rangle-\langle A_1' A_2' A_3'\rangle|\leqslant 1, \tag{12}$$

which is the same as the corresponding Bell's inequality derived through the use of the assumption of locality.[8-10] The same comment applies to Bell's inequalities for arbitrary N: Since (10) is the same as the corresponding expression derived from the concepts of the local theory of hidden parameters,[9,10] local Bell's inequalities can be generalized to nonlocal ones by the algorithm described here.

We recall[9-11] that Bell's inequalities for $N>2$ attract interest because of the quantitative growth of the discrepancies with the predictions of quantum theory: by a factor of $2^{(N-1)/2}$. For example, the left side of (12) takes on a value of 2 in a quantum treatment under certain conditions. Furthermore, starting with $N=3$ one can formulate the Greenberger–Horn–Zeilinger $+1=-1$ paradox in a graphic way.[8-10,12] It assumes a complete correlation between the results of measurements, e.g.,

$$\langle A_1' A_2 A_3\rangle=\langle A_1' A_2 A_3\rangle=\langle A_1 A_2' A_3\rangle=A_1 A_2' A_3=\langle A_1 A_2 A_3'\rangle$$
$$=A_1 A_2 A_3'=-\langle A_1' A_2' A_3'\rangle=-A_1' A_2' A_3'=1, \tag{13}$$

which quantum theory allows. We thus have a product

$$(A_1' A_2 A_3)(A_1 A_2' A_3)(A_1 A_2 A_3')(A_1' A_2' A_3')=-1. \tag{14}$$

It is shown below that this result is not possible under requirements (2)–(4).

Under conditions of a complete correlation, (13), the components W_M, $M=1, 2, ...64$, which contribute $A'_{M1} A_{M2} A_{M3}=-1$, must be zero. The same comment applies to the components which contribute $A_{M1} A'_{M2} A_{M3}=A_{M1} A_{M2} A'_{M3}=-1$. As a result, of the 64 we are left with only 8 nonzero components W_M, but all of them contribute $A'_{M1} A'_{M2} A'_{M3}=+1$! Furthermore, there is no component W_M which would contribute identical signs in any three of the four factors in the parenthetical expression on the left side of (14), while contributing the opposite sign in the fourth factor. The reason is the product

$$(A'_{M1}A_{M2}A_{M3})(A_{M1}A'_{M2}A_{M3})(A_{M1}A_{M2}A'_{M3})(A'_{M1}A'_{M2}A'_{M3})$$
$$=(A_{M1}A'_{M1}A_{M2}A'_{M2}A_{M3}A'_{M3})^2=+1. \tag{15}$$

Consequently, if restrictions (2)–(4) do allow the complete correlation required under these conditions, then they do so only in the case in which product (14) is $+1$. The "nonlocal" Greenburger–Horn–Zeilinger paradox formulated here does not require the hypothesis of locality, so the rejection of the locality hypothesis does not resolve that paradox. These arguments can easily be generalized to arbitrary $N\geqslant 3$.

The algorithm proposed here for proving Bell's inequalities also allows one to generalize the Bell's inequalities discussed here to the case of nondichotomic variables which lie in the interval $[-1,+1]$. The number of components M increases in this case, but (11) remains valid by virtue of the condition $|S_{MN}|\leqslant 1$. Bell's inequalities should therefore hold under conditions (2)–(4).

Just why is it that the relations formulated here do not correspond to the quantum-mechanical results of experiments, which are possible in principle, which would be capable of demonstrating violations of Bell's inequalities or the Greenberger–Horn–Zeilinger paradox? The *sole* argument which explains these contradictions is that a positive definite probability distribution does not exist in such cases. If, within the framework of quantum theory, we calculate joint probabilities W_M, which are equal to the corresponding moments of discrete variables (1), then for a specific model of an optical interference experiment,[13,14] in which violations of Bell's inequalities are demonstrated, we find (see also Ref. 7)

$$W(A_1,A'_1,A_2,A'_2)=(1/16)[1+A_1A'_1\cos(\alpha-\alpha')+A_2A'_2\cos(\beta-\beta')$$
$$+A_1A'_1A_2A'_2\cos(\alpha+\beta-\alpha'-\beta')+A_1A_2\cos(\alpha+\beta)$$
$$+A'_1A_2\cos(\alpha'+\beta)+A_1A'_2\cos(\alpha+\beta')+A'_1A'_2\cos(\alpha'+\beta')]. \tag{16}$$

Here α and β are the phase delays in the channels of the two observers ($N=2$) in the regime in which they detect the variables A_1 and A_2, while α' and β' correspond to the regimes in which the "primed" variables are detected. Let us assume $\alpha=0$, $\alpha'=\pi/2$, $\beta=-\pi/4$, and $\beta'=\pi/4$. According to (16), some of the joint probabilities are then negative,

$$W(++--)=W(+-+-)=W(-+-+)=W(--++)=-2^{1/2}/16, \tag{17}$$

and the left side of (5) turns out to be equal to $2^{1/2}$.

The *sole* reason why (5) does not hold under the hypothesis of the existence of a probability distribution $W(A_1A'_1A_2A'_2)$ is thus the violation of (2), since, according to (16), normalization conditions (3) and correspondence conditions (4) hold.

The probability distribution $W(A,A',B,B')$ is analogous to a Wigner distribution function: Some of the observables in it ("primed" and "unprimed") are described by noncommuting operators. Accordingly, this distribution can take on negative values.

It can thus be concluded that rejecting the hypothesis of locality does not resolve these paradoxes.

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