Sustenance of vortex structures in a rotating fluid layer heated from below

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A system of thermohydrodynamic equations describing the evolution of large-scale pertubations in a rapidly rotating layer of fluid, heated from below, is analyzed in the Boussinesq approximation. The incorporation of viscosity, the deformation of the free upper surface, and the heating leads to new terms in the equation. These new terms represent positive and negative diffusion. One solution of the equation corresponds to localized Rossby vortices. There is an amplitude increase for perturbations of sufficiently large scale. A quasisteady solution is derived for a nonlinear 1D equation by numerical calculations.

1. Among the important advances toward resolving the fundamental problem of the long lifetime of the great red spot of Jupiter are models in which the spot is associated with a solitary vortex.[1] In addition to the nonlinear effects which make possible the existence of a localized structure, there must be some mechanism to compensate for the dissipative loss, since the incorporation of dissipation results in the decay of the vortex over a time on the order of three years (Ref. 2). At present, the most popular point of view is that the loss is replenished by energy supplied from the unstable zonal shear flow against whose profile the spot is observed.[2] However, since over the more than three-hundred years of observations the dynamic characteristics of the Jovian atmosphere have varied rather significantly,[3] it is difficult to rule out the possibility that this mechanism cuts off in certain time intervals. We would also like to have a pumping mechanism based on some constant properties, which do not depend on the time. As we show below in the example of the simplest model, the role of such a property may be played by one of the important distinguishing features of Jupiter: the heating of the atmosphere from lower-lying layers.[4,5]

2. Let us consider a rotating layer of liquid which is heated from below and which is unbounded horizontally. The equations are written in the Boussinesq approximation. The boundary conditions differ from the classical Rayleigh formulation[6] in the following way: It is assumed that the lower surface is undeformable and that the upper surface is a deformable free surface. The surfaces are held at constant temperatures. The pressure at the upper boundary is zero, while at the lower boundary the derivative of the pressure along the vertical coordinate is zero, by virtue of our assumption that there is no tangential stress. We consider the case of rapid rotation, in which the square root of the reciprocal of the Taylor number is a small parameter of the problem, $\mathrm{Ta}^{-1/2} \ll 1$. Carrying out a quasigeostrophic expansion, we incorporate the viscous terms, along with the nonlinear advection terms. We must do this in order to satisfy the relation among the horizontal component of the velocity V, the kinematic viscosity coefficient ν, and the horizontal scale L: $V \sim \nu/L$. For sufficiently large horizontal length scales of the flows, at $L \gg 1$, heat diffusion is incorporated in the heat-conduction equation only along the vertical coordinate. The approximation of a beta plane is used to model flows on a sphere. As a result, we find the following system of equations, which differs from that ordinarily used in geophysical hydrodynamics[7] in that it contains diffusion terms:

$$w_z - \frac{1}{D^2}\Delta_2 p_t - \frac{1}{PD^3} J(p, \Delta_2 p) - \frac{\beta}{D} D p_x + \frac{1}{D^2}\Delta_2^2 p + \frac{1}{D^2}\Delta_2 p_{zz} = 0, \tag{1}$$

$$p_z = RT, \tag{2}$$

$$RT_t + u_0 T_x + v_0 T_y - w = T_{zz}, \tag{3}$$

$$u_0 = -\frac{1}{D} p_y, \quad v_0 = \frac{1}{D} p_x, \tag{4}$$

$$\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad J(f, g) = f_x g_y - f_y g_x.$$

Here p is the deviation from the unperturbed (in the absence of a large-scale deformation or temperature) linear hydrostatic profile, w is the vertical velocity component, u_0 and v_0 are the geostrophic horizontal velocity components, and T is the deviation of the temperature from the linear hydrostatic profile found in the absence of a deformation of the surface. We switch to dimensionless units as in Ref. 6. Here R is the Rayleigh number, P is the Prandtl number, $\mathrm{Ta} = D^2$ is the Taylor number, and β is a parameter characterizing the beta effect: the latitudinal variation of the angular rotation velocity.

We write the boundary conditions on the perturbed quantities as follows:

$$T = 0, \; p_z = 0, \quad w = 0 \text{ at } z = 0; \tag{5}$$

$$T = h, \; p = qh, \; w = Ph_t + u_0 h_x + v_0 h_y \text{ at } z = 1 + h. \tag{6}$$

Here q is a parameter characterizing the depth of the liquid, and h is the deviation of the surface from its unperturbed state.

The solution of this system of equations is the sum of a barotropic component (with a pressure distribution which does not depend on the vertical coordinate) and a baroclinic component. We consider the case in which the baroclinic component can be ignored everywhere in Eq. (1) except in the last term (quasibarotropic flows). A necessary condition here is $R \ll q$. From the condition that the last baroclinic term be retained along with the next-to-last diffusion term we find $R \sim q/L^2$. These relations hold at $L \gg 1$. In other words, they actually do not introduce the further limitations which are used in a quasigeostrophic expansion.

The solution of the static heat-conduction equation with a perturbation $h = \text{const}$, within terms quadratic in h, is $T_{\text{stat}} = zh$. Taking into account the deformation of the free upper surface, we must use $T = zh(x, y)$ as a solution for the temperature. The presence of a deformation leads to the appearance of horizontal temperature gradients. The baroclinic component is a "thermal wind"[7] which is rotating in either a cyclonic or anticyclonic fashion, depending on whether the free surface descends or rises. Substitution of the solution into the heat-conduction leads to an expression $w=z(Ph_t+u_0 h_x+v_0 h_y)$ for the vertical component of the velocity. This result agrees with boundary conditions (5) and (6).

Integrating Eq. (1) along the coordinate z, taking account of the baroclinic component of the pressure only in the last term, and taking account of the quadratic nonlinearity in h in the β term, we find the equation

$$Ph_t-\frac{q}{D^2}\Delta_2 h_t-\frac{q^2}{PD^3}J(h,\Delta_2 h)-\frac{q}{D}\beta(1+h)h_x+\frac{q}{D^2}\Delta_2^2 h+\frac{R}{D^2}\Delta_2 h=0. \quad (7)$$

This equation differs from the Obukhov–Charni equation in that diffusion terms with positive and negative diffusion coefficients have been added, and the nonlinearity in h in the β term has been taken into account. The solution of the equation after a linearization in terms of h yields the critical value of the wave vector,

$$k_{\rm cr}=(R/q)^{1/2}, \quad (8)$$

which forms the boundary between growing and decaying modes.

The excitation of a barotropic wave which deforms the free upper surface in a layer of liquid heated from below gives rise to a baroclinic component, which is capable of intensifying a deformation.

We substitute the value of the bulk expansion coefficient α for Jupiter,[9] $\alpha\sim 6\times 10^{-3}(1/K)$, into Eq. (8) for the critical value of the wave number, $k_{\rm cr}=(\alpha\Delta T)^{1/2}$. The temperature difference between the upper and lower surfaces, ΔT, is estimated to be $\Delta T\sim 10$ K. The relation for the scale at which amplification begins becomes $L\sim 1000$ km. This figure is much smaller than the size of the great red spot of Jupiter, $\sim 10\ 000$ km. This estimate thus supports the suggestion that the spot may be sustained by the mechanism which has been found for the amplification of vortex perturbations.

3. If the shape of the spot is to be maintained in a steady state, there is the further requirement that the pumping be balanced by a dissipation and not lead to a disruption of a localized vortex. Because of the long-wave instability inherent in Eq. (7), there can be no exact steady-state localized solutions: A negative diffusion will lead to a growth of noisy perturbations. If the pumping is sufficiently weak, however, and if the growth time of the noisy perturbations is sufficiently long, a vortex whose amplitude and shape do not change substantially during the growth of the perturbations can be regarded as a quasi-steady-state structure.

In the dissipationless case, the nonlinearity described by the fourth term in Eq. (7) plays a governing role in maintaining the localization of large-scale vortices.[2] The role of this nonlinearity in establishing the balance required to maintain the quasi-steady amplitude and shape of the vortex can be studied most simply in this case in the example of the 1D version of Eq. (7):

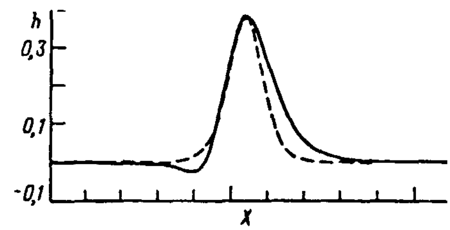

FIG. 1. Solid curve—Steady-state solution of the 1D equation with positive and negative diffusion; dashed curve—solition solution of the regularized long-wave equation of the same amplitude. $R=500$, $\mathrm{Ta}=10^4$, $q=100$, $\beta=2$, $P=1$. The arrow shows the drift direction.

$$Ph_t-\frac{q}{D^2}h_{xxt}-\frac{q}{D}\beta(1+h)h_x+\frac{q}{D^2}h_{xxxx}+\frac{R}{D^2}h_{xx}=0. \tag{9}$$

On the one hand, this equation is a generalization of the Kuramoto–Sivashinsky equation[10,11] (a dispersive term h_{xxt} is added); on the other, it is a generalization of the regularized long-wave equation (terms with positive and negative diffusion are added).[12] Numerical calculations have been carried out for Eq. (9). A periodic boundary condition corresponding to the traversal of a circle of a sphere was used along the horizontal coordinate. An initial perturbation was specified in the form of an elevation with an amplitude amounting to 0.1 of the thickness of the layer and constituting a steady-state soliton solution of the equation without the diffusion terms.[13] The time evolution of the perturbation was then calculated. When a zero Rayleigh number was specified, a decay of the perturbation was observed over a time on the order of 5 dimensionless units. When a nonzero Rayleigh number was specified, the amplitude of the perturbation increased. At the same time, the crest of the pulse became sharper, and its leading edge steeper, by virtue of a nonlinearity. As a result, the role played by the term with a positive diffusion increased, and the increase in the amplitude came to a halt. The elevation reached a steady-state shape after a time on the order of 100 dimensionless units. Figure 1 shows the solution found. These calculations demonstrate that the amplitude and shape of the elevation are preserved for a time on the order of 150 units—up to the point at which there is a substantial increase in the noisy perturbations. One might suggest that, again in the 2D case, this nonlinearity would lead to the establishment of a finite vortex amplitude. When the pumping mechanism discussed here is applied to the great red spot of Jupiter, one must also bear in mind that the presence of zonal flows might contribute to the localization and uniqueness of the spot. Such flows would smooth out noisy perturbations and give rise to an effective threshold amplitude, at which growth due to the long-wave instability would begin. This work had support from the Russian Basic Research Foundation (Project 93-02-17014).

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