Quantization of systems based on the odd Poisson bracket

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New quantum representations of the odd (with respect to the Grassmann grading) Poisson bracket are obtained. Some of them are responsible, as demonstrated by a simple example, for the quantization of the classical Hamiltonian dynamics based on the odd Poisson bracket.

1. A few years ago, the prescription[1] for the canonical quantization of the odd Poisson bracket (the latter has naturally appeared in the Batalin–Vilkovisky scheme[2] for the quantization of the gauge theories) was suggested, and several odd-bracket quantum representations for the canonical variables were obtained. In contrast with the even Poisson bracket, some of the odd-bracket quantum representations turned out to be not equivalent.[3] Later, it was revealed that an odd Poisson bracket is responsible for the description of the dynamics of some Hamiltonian systems.[4] Specifically, for the systems having an equal number of pairs of even and odd (relative to the Grassmann grading) phase coordinates it was proved that the Hamilton equations of motion obtained by means of an even Poisson bracket with the help of an even Hamiltonian can be reproduced by the odd bracket using the equivalent odd Hamiltonian. However, the direct connection of the odd-bracket quantum representations for the canonical variables with the quantization of the classical Hamiltonian dynamics based on the odd Poisson bracket has not been formulated explicitly until now. Having two equivalent ways of describing the classical dynamics of the above-mentioned systems,[4] we can try to find such odd-bracket quantum representations which with use of the classically equivalent odd Hamiltonian will give a quantum description of the systems which coincides with that obtained from the corresponding even Hamiltonian with use of the even-bracket quantum representations. In other words, the equivalence of the description of dynamics with the brackets of different Grassmann parities can be extended from the classical level to the quantum level.

In the present letter we introduce new odd-bracket quantum representations which extend those obtained in Refs. 1 and 3. At the end of the paper we demonstrate by using a simple example of the supersymmetric one-dimensional oscillator that among these representations there are only those relevant to the quantization of the classical Hamiltonian systems, whose dynamics is formulated by means of the odd bracket with the help of an odd Hamiltonian.

2. First, we recall the necessary properties of various graded Poisson brackets. The even and odd brackets in terms of the real even $y_i=(q^a,p_a)$ and odd $\eta^i=\theta^\alpha$ canonical variables have, respectively, the form

$$\{A,B\}_0=A\left\{\sum_{a}^{n}(\overleftarrow{\partial}_{q^\epsilon}\overrightarrow{\partial}_{p_\epsilon}-\overleftarrow{\partial}_{p_\epsilon}\overrightarrow{\partial}_{q^\epsilon})-i\sum_{\alpha=1}^{2m}\overleftarrow{\partial}_{\theta^\alpha}\overrightarrow{\partial}_{\theta^\alpha}\right\}B; \tag{1}$$

$$\{A,B\}_1=A\sum_{i=1}^{N}(\overleftarrow{\partial}_{y_i}\overrightarrow{\partial}_{\eta^i}-\overleftarrow{\partial}_{\eta^i}\overrightarrow{\partial}_{y_i})B, \tag{2}$$

where $g(A)$ is the Grassmann grading of the quantity A, $\overleftarrow{\partial}$ and $\overrightarrow{\partial}$ are the right and left derivatives, and we introduce the notation $\partial_x=\partial/\partial x$. By introducing, in addition to the Grassmann grading $g(A)$ of any quantity A, its corresponding bracket grading $g_\epsilon(A)=g(A)+\epsilon \pmod 2$ ($\epsilon=0, 1$), the grading and symmetry properties, the Jacobi identities, and the Leibnitz rule can be uniformly expressed for both brackets (1,2) as

$$g_\epsilon(\{A,B\}_\epsilon)=g_\epsilon(A)+g_\epsilon(B) \pmod 2, \tag{3}$$

$$\{A,B\}_\epsilon=-(-1)^{g_\epsilon(A)g_\epsilon(B)}\{B,A\}_\epsilon, \tag{4}$$

$$\sum_{(ABC)}(-1)^{g_\epsilon(A)g_\epsilon(C)}\{A,\{B,C\}_\epsilon\}_\epsilon=0, \tag{5}$$

$$\{A,BC\}_\epsilon=\{A,B\}_\epsilon C+(-1)^{g_\epsilon(A)g(B)}B\{A,C\}_\epsilon, \tag{6}$$

where (3)–(5) have the shape of the Lie superalgebra relations in their canonical form.[5] Here $g_\epsilon(A)$ is the canonical grading for the corresponding bracket.

3. The procedure of the odd-bracket canonical quantization, given in Refs. 1 and 3, involves splitting all the canonical variables into two sets, dividing all the functions which depend on the canonical variables into classes, and introducing the quantum multiplication $*$, which is either the common product or the bracket composition, in the dependence on the classes to which the co-factors belong. One of the classes in this case must contain the normalized wave functions, and the result of the multiplication $*$ for any quantity on the wave function Ψ must belong to the class that contains Ψ. This procedure is the generalization in the odd-bracket case of the canonical quantization rules for the usual Poisson bracket $\{...,...\}_{\text{Pois.}}$, which, for example, in the coordinate representation for the canonical variables q and p is defined as

$$q*\Psi(q)=q\Psi(q),\quad p*\Psi(q)=i\hbar\{p,\Psi(q)\}_{\text{Pois.}}=-i\hbar\frac{\partial\Psi}{\partial q},$$

where $\Psi(q)$ is the normalized wave function which depends on the coordinate q.

In Refs. 1 and 3 two nonequivalent odd-bracket quantum representations for the canonical variables were obtained by using two different methods of dividing the function. These methods, however, do not exhaust all the possibilities. In the present paper we propose a more general division method which contains as the limiting cases those given in Refs. 1 and 3.

Let us construct quantum representations for an arbitrary graded bracket under its canonical quantization. To this end, all canonical variables are split into two sets equal in number, so that none of them should contain pairs of canonical conjugates. Note that to make such a splitting possible for the even bracket (1), the transition must occur from the real, canonical, self-conjugate, odd variables to some pairs of odd variables, which simultaneously are complex and canonical conjugate to each other. Forming from the integer degrees of the variables of one set (we call it the first set) the monomials of the odd $2s+1$ and even $2s$ uniformity degrees and multiplying them by the arbitrary functions which depend on the variables of the other (second) set, we divide all the functions of the canonical variables into the classes designated as $\overset{\epsilon}{O}_s$ and $\overset{\epsilon}{E}_s$, respectively. In general, the odd-bracket canonical variables, for example, can be split. The first set would therefore contain the even y_i $(i=1,..., n<N)$ and the odd $\eta^{n+\alpha}$ $(\alpha=1,..., N-n)$ variables, while the second set would contain the remaining variables. The classes of the functions obtained under this splitting will then have the form

$$\overset{1}{O}_s=(y_i,\eta^{n+\alpha})^{2a+1}f(\eta^i,y_{n+\alpha}); \quad \overset{1}{E}_s=(y_i,\eta^{n+\alpha})^{2s}f(n^i,y_{n+\alpha}),$$

where the factors in front of the arbitrary function $f(\eta^i, y_{n+\alpha})$ are the monomials which have the uniformity degrees indicated in the exponents. These classes satisfy the corresponding bracket relations

$$\{\overset{\epsilon}{O}_s,\overset{\epsilon}{O}_{s'}\}_\epsilon=\overset{\epsilon}{O}_{s+s'}; \quad \{\overset{\epsilon}{O}_s,\overset{\epsilon}{E}_{s'}\}_\epsilon=\overset{\epsilon}{E}_{s+s'}; \quad \{\overset{\epsilon}{E}_s,\overset{\epsilon}{E}_{s'}\}_\epsilon=\overset{\epsilon}{O}_{s+s'-1} \tag{7}$$

and the relations of the ordinary Grassmann multiplication

$$\overset{\epsilon}{O}_s\cdot\overset{\epsilon}{O}_{s'}=\overset{\epsilon}{E}_{s+s'+1}; \quad \overset{\epsilon}{O}_s\cdot\overset{\epsilon}{E}_{s'}=\overset{\epsilon}{O}_{s+s'}; \quad \overset{\epsilon}{E}_s\cdot\overset{\epsilon}{E}_{s'}=\overset{\epsilon}{E}_{s+s'}. \tag{8}$$

It follows from (7) and (8) that $\overset{\epsilon}{O}=\{\overset{\epsilon}{O}_s\}$ and $\overset{\epsilon}{E}=\{\overset{\epsilon}{E}_s\}$ form a superalgebra with respect to the addition and the quantum multiplication $*_\epsilon$ $(\epsilon=0,1)$, defined for the corresponding bracket as

$$\overset{\epsilon'}{O}*_\epsilon\overset{\epsilon''}{O}=\{\overset{\epsilon'}{O},\overset{\epsilon''}{O}\}_\epsilon\in\overset{\epsilon}{O}; \quad \overset{\epsilon'}{O}*_\epsilon\overset{\epsilon''}{E}=\{\overset{\epsilon'}{O},\overset{\epsilon''}{E}\}_\epsilon\in\overset{\epsilon}{E}; \quad \overset{\epsilon'}{E}*_\epsilon\overset{\epsilon''}{E}=\overset{\epsilon'}{E}\cdot\overset{\epsilon''}{E}\in\overset{\epsilon}{E}, \tag{9}$$

where $\overset{\epsilon'}{O},\overset{\epsilon''}{O}\in\overset{\epsilon}{O}$ and $\overset{\epsilon'}{E},\overset{\epsilon''}{E}\in\overset{\epsilon}{E}$. Note that the classes $\overset{\epsilon}{O}_0$ and $\overset{\epsilon}{E}_0$ form the sub-superalgebra. In terms of the quantum grading $q_\epsilon(A)$ of any quantity A

$$q_\epsilon(A)=\begin{cases} g_\epsilon(A), & \text{for } A\in\overset{\epsilon}{O}; \\ g(A), & \text{for } A\in\overset{\epsilon}{E}, \end{cases}$$

which is introduced for the appropriate bracket, the grading, and the symmetry properties of the quantum multiplication $*_\epsilon$ arising from the corresponding properties of the bracket (3,4) and the Grassmann composition of any two quantities A and B are uniformly written as

$$q_\epsilon(A*_\epsilon B)=q_\epsilon(A)+q_\epsilon(B) \pmod 2 \tag{10a}$$

$$\overset{\epsilon'}{O}*_\epsilon\overset{\epsilon''}{O}=-(-1)^{g_\epsilon(\overset{\epsilon'}{O})g_\epsilon(\overset{\epsilon''}{O})}\overset{\epsilon''}{O}*_\epsilon\overset{\epsilon'}{O}, \tag{10b}$$

$$\overset{\epsilon'}{E}*_\epsilon\overset{\epsilon''}{E}=(-1)^{g_\epsilon(\overset{\epsilon'}{E})g_\epsilon(\overset{\epsilon''}{E})}\overset{\epsilon''}{E}*_\epsilon\overset{\epsilon'}{E}, \tag{10c}$$

Using the quantum multiplication $*_\epsilon$ and the quantum grading q_ϵ, we define for any two quantities A and B the quantum bracket [(anti)commutator] $[A,B\}_\epsilon$ (under its action on the wave function Ψ which is assumed to belong to the class E (Refs. 1 and 3) in the form

$$[A,B\}_\epsilon *_\epsilon \Psi = A *_\epsilon (B *_\epsilon \Psi) - (-1)^{g_\epsilon(A) g_\epsilon(B)} B *_\epsilon (A *_\epsilon \Psi). \tag{11}$$

If $A,B \in \overset{\epsilon}{E}$, then, due to (10c), the quantum bracket between them equals zero. In particular, the wave functions are (anti)commutative. If A or the two quantities A and B belong to the class $\overset{\epsilon}{O}$, then in the first case, due to the Leibnitz rule (6), and in the second case, because of the Jacobi identities (5), we obtain the following relation from the definitions (10) and (11):

$$[A,B\}_\epsilon *_\epsilon \Psi = \{A,B\}_\epsilon *_\epsilon \Psi = (A *_\epsilon B) *_\epsilon \Psi.$$

This relation establishes a connection between the classical bracket and the quantum bracket of the corresponding Grassmann parity. Note that the quantization procedure admits a reduction to $O_0 \cup E_0$.

The grading q_ϵ determines the symmetry properties of the quantum bracket (11). Under above-mentioned splitting of the odd-bracket canonical variables into two sets, the grading q_1 equals unity for the variables $y_i \in \overset{1}{O}$ and $\eta^i \in \overset{1}{E}$ $(i=1,\dots, n \leqslant N)$ and is equal to zero for the remaining canonical variables $y_{n+\alpha} \in \overset{1}{E}, \eta^{n+\alpha} \in \overset{1}{O}$ $(\alpha = 1,\dots, N - n)$. In this case, therefore, the quantum odd bracket is represented with the anticommutators between the quantities y_i and η^i and with the commutators for the remaining relations of the canonical variables. If the roles of the first and second sets of the canonical variables change, then the quantum bracket is represented with the anticommutators between $y_{n+\alpha}$ and $\eta^{n+\alpha}$ and with the commutators in the other relations. In Refs. 1 and 3 the odd-bracket quantum representations were obtained for the cases $n=0$, N which contain, respectively, only the commutators or anticommutators.

4. As a simple example of using the odd-bracket quantum representations under the quantization of the classical systems based on the odd bracket, let us consider the one-dimensional supersymmetric oscillator, whose phase superspace x^A contains a pair of even q, p and a pair of odd η^1, η^2 real canonical coordinates. In terms of more suitable complex coordinates $z = (p - iq)/\sqrt{2}$, $\eta = (\eta^1 - i\eta)/\sqrt{2}$ and their complex conjugates $\bar{z}$ and $\bar{\eta}$, the even bracket is written as

$$\{A,B\}_0 = iA[\overleftarrow{\partial}_{\bar{z}}\overrightarrow{\partial}_z - \overleftarrow{\partial}_z\overrightarrow{\partial}_{\bar{z}} - (\overleftarrow{\partial}_{\bar{\eta}}\overrightarrow{\partial}_\eta - \overleftarrow{\partial}_\eta\overrightarrow{\partial}_{\bar{\eta}})]B \tag{12}$$

and the even Hamiltonian H, the supercharges Q_1 and Q_2, and the fermionic charge F are

$$H = z\bar{z} + \bar{\eta}\eta; \quad Q_1 = \bar{z}\eta + z\bar{\eta}; \quad Q_2 = i(\bar{z}\eta - z\bar{\eta}); \quad F = \eta\bar{\eta}. \tag{13}$$

The odd Hamiltonian H and the appropriate odd bracket, which reproduce the same Hamiltonian equations of motion as those resulting from (12) with an even Hamiltonian H (13), i.e., which satisfy the condition[4,6]

$$\frac{dx^A}{dt}=\{x^A,H\}_0=\{x^A,\bar{H}\}_1 \tag{14}$$

(t is the time), can be assumed to be $\bar{H}=Q_1$ and

$$\{A,B\}_1=iA(\overleftarrow{\partial}_{\bar{z}}\vec{\partial}_\eta-\overleftarrow{\partial}_\eta\vec{\partial}_{\bar{z}}+\overleftarrow{\partial}_{\bar{\eta}}\vec{\partial}_z-\overleftarrow{\partial}_z\vec{\partial}_{\bar{\eta}})B. \tag{15}$$

The complex variables have the advantage over the real variables, because with their use the splitting of the canonical variables into two sets $\bar{z},\bar{\eta}$ and z,η satisfies simultaneously the requirements necessary for the quantization of the brackets, (12) and (15). Any one of the vector fields $\overset{\epsilon}{X}_{A_i} = -i\{A_i,...\}_\epsilon$ for the quantities $\{A_i\}=(H,Q_1,Q_2,F)$, which describes the dynamics and the symmetry of the system under consideration, is split into the sum of two differential operators which depend on either $\bar{z},\bar{\eta}$ or z,η. From (12)–(15), for example, we have

$$\overset{0}{X}_H=\overset{1}{X}_H=z\partial_z+\eta\partial_\eta-\bar{z}\partial_{\bar{z}}-\bar{\eta}\partial_{\bar{\eta}}. \tag{16}$$

The diagonalization does not take place in terms of the variables $x^A=(q,p;\eta^1,\eta^2)$.

In accordance with the above-mentioned splitting of the complex variables, we can perform one of the two possible divisions of all the functions into the classes which are common for the two brackets, (12) and (15). They play a crucial role under their canonical quantization and lead to the same quantum dynamics for the system under consideration. If $\bar{z}$ and $\bar{\eta}$ are attributed to the first set, then the corresponding function division is

$$\overset{\epsilon}{O}_s=(\bar{z}\bar{\eta})^{2s+1}f(z,\eta);\quad \overset{\epsilon}{E}_s=(\bar{z}\bar{\eta})^{2s}f(z,\eta).$$

If we restrict the analysis to the classes O_0 and E_0, then $\Psi\in E_0$ and depends only on z,η and $A_i\in O_0$. According to the definition (9), the results of the quantum multiplications $*_1$ and $*_0$ of $z,\eta\in E_0$ and $\bar{z},\bar{\eta}\in O_0$ on the wave function Ψ are

$$\begin{aligned}&z*_1\Psi=z*_0\Psi=z\cdot\Psi;\quad \bar{\eta}*_1\Psi=\bar{z}*_0\Psi=\partial_z\Psi;\\&\eta*_1\Psi=\eta*_0\Psi=\eta\cdot\Psi;\quad \bar{z}*_1\Psi=-\bar{\eta}*_0\Psi=\partial_\eta\Psi.\end{aligned} \tag{17}$$

The positive definite scalar product of the wave functions $\Psi_1(z,\eta)$ and $\Psi_2(z,\eta)$ can be determined in the form[7]

$$(\Psi_1,\Psi_2)=\frac{1}{\pi}\int\exp[$$

$$=-(|z|^2+\bar{\theta}\eta)]\Psi_1(z,\eta)[\Psi_2(z,\theta)]+d\bar{\theta}\,d\eta\,d(\mathrm{Re}\,z)d(\mathrm{Im}\,z), \tag{18}$$

where θ is the auxiliary complex Grassmann quantity which anticommutes with η, and the integration over the real and imaginary components of z is performed in the limits $(-\infty,\infty)$. It is easy to see that with respect to the scalar product (18) the pairs of the canonical variables, Hermitian conjugated to each other under the multiplication $*_1$, are $z,\bar{\eta}$ and $\bar{z},\eta$, but under $*_0$ are z, $\bar{z}$ and $\eta,-\bar{\eta}$.

In order to have the Hamiltonian operator, which is obtained from the system quantization, act on the wave function, it is necessary, as we know, to replace the canonical variables in the classical Hamiltonian by the respective operators or, equivalently, define their action with the help of the corresponding quantum multiplication $*_\epsilon$. In this connection, in view of (16) and (17), we see that the self-consistent quantum Hamiltonian operators in the even and odd cases, which are in agreement with the classical expressions (13) for the equivalent Hamiltonians H and $\bar{H}$ and which give the same result at the action on $\Psi(z,\eta)$, are, respectively,

$$H*_0\Psi = z*_0(\bar{z}*_0\Psi) - \eta*_0(\bar{\eta}*_0\Psi); \tag{19}$$

$$\bar{H}*_1\Psi = z*_1(\bar{\eta}*_1\Psi) + \eta*_1(\bar{z}*_1\Psi). \tag{20}$$

The Hamiltonians (19) and (20) are Hermitian relative to the scalar product (18) and, due to (17), are reduced to the Hamiltonian operator for the one-dimensional supersymmetric oscillator $H = a^+a + b^+b$, which is expressed in terms of the creation and annihilation operators for the bosons $a^+ = z$, $a = \partial_z$ and fermions $b^+ = \eta$, $b = \partial_\eta$, respectively, in the Fock–Bargmann representation (see, for example, Ref. 8). The normalized, with respect to (18), eigenfunctions $\Psi_{k,n}(z,\eta)$ of the Hamiltonians (19) and (20), which correspond to the energy eigenvalues $E_{k,n} = k + n$ ($k = 0, 1$; $n = 0, 1, \ldots, \infty$), have the form

$$\Psi_{k,n}(z,\eta) = \frac{1}{\sqrt{n!}} (\eta*_\epsilon)^k \ (z*_\epsilon)^n 1.$$

Note that another equivalent representation of the quantum supersymmetric oscillator can be obtained, if the canonical variables z and η are chosen as the first set.

5. Thus, we have demonstrated that the use of the quantum representations found for the odd bracket leads to the self-consistent quantization of the classical Hamiltonian systems based on this bracket. We should apparently expect that these representations are applicable for the quantization of more complicated classical systems with the odd bracket.

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