How could you measure a negative probability?

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An algorithm is proposed for reconstructing values of joint probabilities for observables described by noncommuting operators from the results of an interference experiment carried out under photon-counting conditions.

Questions concerning possibilities for finding negative[1–9] or even complex[2,10] values of probabilities have attracted the interest of researchers for many years. The argument here is that a distribution function of random quantities described by noncommuting operators is not always positive definite. A clear example is the Wigner distribution function for the coordinate and momentum of a quantum-mechanical particle. In the orthodox Copenhagen interpretation of quantum mechanics, such probability distributions do not exist, since it would be impossible to carry out an experiment to measure them directly. Nevertheless, indirect measurement methods are possible. This point was demonstrated by Smithey *et al.*,[11] who reconstructed a continuous 2D Wigner distribution of quadrature components of light in a squeezed state. Although negative distributions were not observed there, the quantum theory does not impose any fundamental restrictions of this sort.

In this letter we wish to propose a procedure for reconstructing a multidimensional, discrete probability distribution which takes on negative values, in connection with an experiment to observe an intensity interference,[3,12,13] which would demonstrate Bell's paradox.[14] A schematic diagram of the experiment is shown in Fig. 1. Measurements are carried out in four regimes. The phase delays α and β are established first, then α' and β, then α and β', and finally α' and β'.

The results of the experiment can be described at a formal level by a discrete 4D probability distribution $P^{AA'BB'}_{\alpha\alpha'\beta\beta'}$, which relates dichotomic variables which take on unit values:

$$A=\pm 1, \quad A'=\pm 1, \quad B=\pm 1, \quad B'=\pm 1. \tag{1}$$

If observer A detects, with a phase delay α, the photon count at detector $+$, then this event is assigned a value $A=+1$. If the same event occurs with a phase delay α', then we have $A'=+1$. The photon counts at detector $-$ are coded in a corresponding way, as are the counts in the channels of observer B. The primed variables here correspond to the primed phase delays.

Quantum theory predicts that sign-varying values of $P^{AA'BB'}_{\alpha\alpha'\beta\beta'}$ can be obtained, as can 3D probability distributions of the types $P^{AA'B}_{\alpha\alpha'\beta}$, $P^{AA'B'}_{\alpha\alpha'\beta'}$, etc.[15] Calculating the latter distributions from the experimental results is the goal of the discussion below.

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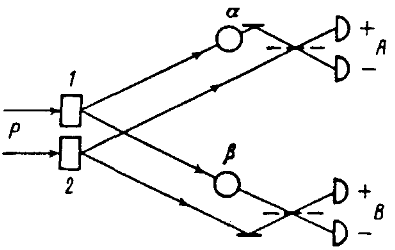

FIG. 1. Schematic diagram of an intensity interferometer with parametric sources of electromagnetic radiation for two observers, A and B. Correlated photons are produced simultaneously in nonlinear elements *1* and *2* by a common coherent pump P. These photons are directed to A and B in two modes, one of which experiences a phase delay (the circles). The modes are mixed at 50% beam splitters (the dashed line segments) and are detected by identical detectors + and −.

We wish to stress that only 2D joint probabilities $P^{AB}_{\alpha\beta}$, $P^{A'B}_{\alpha'\beta}$, $P^{AB'}_{\alpha\beta'}$, and $P^{A'B'}_{\alpha'\beta'}$ and 1D probabilities P^{A}_{α} and P^{B}_{β} can be measured directly.

We can formulate certain properties of $P^{AA'BB'}_{\alpha\alpha'\beta\beta'}$ which will make it possible to solve the inverse problem posed here.

Since the choice of the signs of the variables is arbitrary and is at our disposal if the detectors are identical, the following symmetry must be observed:

$$P^{AA'BB'}_{\alpha\alpha'\beta\beta'} = P^{\overline{AA'BB'}}_{\alpha\alpha'\beta\beta'}, \quad \bar{A} = -A, \quad \bar{A}' = -A', \quad \bar{B} = -B, \quad \bar{B}' = -B'. \tag{2}$$

A consequence of this symmetry is

$$P^{A}_{\alpha} \sum_{A'} \sum_{B} \sum_{B'} P^{AA'BB'}_{\alpha\alpha'\beta\beta'} = \sum_{A'} \sum_{B} \sum_{B'} P^{\overline{AA'BB'}}_{\alpha\alpha'\beta\beta'} = P^{\bar{A}}_{\alpha} = P^{B}_{\beta} = P^{\bar{B}}_{\beta} = 1/2, \tag{3}$$

which agrees with the experimental results of Ref. 13. Here we have used correspondence conditions and probability normalization conditions.

By virtue of the symmetry of the primed and unprimed variables, we have

$$P^{AA'BB'}_{\alpha\alpha'\beta\beta'} = P^{AA'BB'}_{\alpha'\alpha\beta'\beta}. \tag{4}$$

Since the experiment is of an interference nature, the dependence of $P^{AA'BB'}_{\alpha\alpha'\beta\beta'}$ on one of the phase delays must be a harmonic dependence of the type

$$P^{AA'BB'}_{\alpha\alpha'\beta\beta'} = G(\alpha',\beta,\beta') + H(\alpha',\beta,\beta')\cos[\alpha + \varphi(\alpha',\beta,\beta')]. \tag{5}$$

This relation also holds for probability distributions of lower dimensionalities. This relation follows from the classical stochastic description of the mixing of interfering waves at beam splitters. The doubled intensity of the electromagnetic radiation at the + detector of observer A is

$$2n_+^a = |a_1 e^{-i\alpha} + a_2|^2$$

$$= |a_1| \exp(-i\alpha - i\varphi_1) + |a_2| \exp(-i\varphi_2)|^2$$

$$= |a_1|^2 + |a_2|^2 + 2|a_1 a_2| \cos(\alpha + \varphi_1 - \varphi_2), \tag{6}$$

where a_j are the complex amplitudes of the signals being mixed, and φ_j are their phases. Correspondingly, for the $-$ detector we have

$$2n_-^a = |a_1|^2 + |a_2|^2 - 2|a_1 a_2| \cos(\alpha + \varphi_1 - \varphi_2). \tag{7}$$

These expressions correspond to (5) if we assume that the joint probabilities are proportional to the intensities. Expression (5) is also consistent with quantum theory.

For the same reasons we can write

$$P_{\alpha\alpha'\beta\beta'}^{\bar{A}A'BB'} = P_{(\alpha\pm\pi)\alpha'\beta\beta'}^{AA'BB'}, \tag{8}$$

as follows from expressions (6) and (7), since the addition of $\pm\pi$ to α is equivalent to an interchange of n_+^a and n_-^a. Corresponding assertions can be made for the other variables and the corresponding phase delays.

We first reconstruct the joint 2D probability:

$$P_{\alpha\alpha'}^{++} \equiv P_{\alpha\alpha'}^{A=+1, A'=+1}. \tag{9}$$

The correspondence condition gives us

$$P_\alpha^+ = P_{\alpha\alpha'}^{++} + P_{\alpha\alpha'}^{+-}. \tag{10}$$

We assume $\alpha = \alpha'$; then we have $P_{\alpha\alpha}^{+-} = 0$, since we have $A \equiv A'$ in this case. Consequently, using (3), we find

$$P_{\alpha\alpha}^{++} = P_\alpha^+ = 1/2. \tag{11}$$

By virtue of (5) we have

$$P_{\alpha\alpha'}^{++} = G(\alpha) + H(\alpha)\cos[\alpha' + \varphi(\alpha)]. \tag{12}$$

According to (8) we have

$$P_{\alpha\alpha}^{+-} = P_{\alpha\alpha+\pi}^{++} = 0. \tag{13}$$

From (11)–(13) we find

$$G(\alpha) + H(\alpha)\cos[\alpha + \varphi(\alpha)] = 1/2, \tag{14}$$

$$G(\alpha) - H(\alpha)\cos[\alpha + \varphi(\alpha)] = 0, \tag{15}$$

and then

$$G(\alpha) = 1/4, \tag{16}$$

$$H(\alpha) = 1/4\cos[\alpha + \varphi(\alpha)]. \tag{17}$$

In the last relation we set

$$\cos[\alpha+\varphi(\alpha)]\neq 0. \tag{18}$$

Substituting (16) and (17) into (12), we find

$$P^{++}_{\alpha\alpha'}=\{1+\cos[\alpha'+\varphi(\alpha)]/\cos[\alpha+\varphi(\alpha)]\}/4\equiv P^{++}_{\alpha'\alpha}. \tag{19}$$

The last identity is found in accordance with property (4) and is derived by analogy with (3). From (19) we find

$$\varphi(\alpha)\equiv -\alpha\pm m\pi,\quad m=0,1,2,..., \tag{20}$$

which is consistent with (18). Using (8), we thus find

$$P^{AA'}_{\alpha\alpha'}=(1/4)[1+AA'\cos(\alpha-\alpha')]. \tag{21}$$

A formal quantum-mechanical calculation of the moments $\langle n^{a}_{\pm} n^{a'}_{\pm}\rangle$ corresponding to the joint probabilities $P^{\pm\pm}_{\alpha\alpha'}$ leads to the same result.

Three-dimensional probability distributions are related to 2D distributions by simple relations of the type

$$P^{AA'B}_{\alpha\alpha'\beta}=\frac{1}{2}(P^{AB}_{\alpha\beta}+P^{A'B}_{\alpha'\beta}-P^{A\bar{A}'}_{\alpha\alpha'}), \tag{22}$$

as can be verified easily by substituting $P^{AB}_{\alpha\beta}=P^{AA'B}_{\alpha\alpha'\beta}+P^{AA'\bar{B}}_{\alpha\alpha'\beta}$, etc., into the right side of (22).

The joint probabilities $P^{AB}_{\alpha\beta}$ and $P^{A'B}_{\alpha'\beta}$ are determined directly from the experiment; they are

$$P^{AB}_{\alpha\beta}=\frac{1}{4}[1+AB\cos(\alpha+\beta)],\quad P^{A'B}_{\alpha'\beta}=\frac{1}{4}[1+A'B\cos(\alpha'+\beta)]. \tag{23}$$

Substitution of (23) into (22) yields

$$P^{AA'B}_{\alpha\alpha'\beta}=\frac{1}{8}[1+AB\cos(\alpha+\beta)+A'B\cos(\alpha'+\beta)+AA'\cos(\alpha-\alpha')]. \tag{24}$$

A quantum-mechanical calculation of the corresponding moments yields the same result. The joint probabilities which make up this probability distribution can evidently be negative. For example, we would have $P^{+-+}_{\alpha\alpha'\beta}=\frac{1}{8}(1-2^{1/2})$ in the case $\alpha=\pi/2$, $\alpha'=0$, $\beta=\pi/4$.

Working from the experimental data in (23), and assuming properties (2)–(5) and (8), we can thus reconstruct discrete probability distributions of the $P^{AA'B}_{\alpha\alpha'\beta}$ type. These distributions relate observable quantities, some of which are described by noncommuting operators (A and A' in the case at hand). Accordingly, direct measurements of these observables, like measurements of the Wigner distribution, are impossible. In this sense, distributions of this sort are devoid of any working meaning. However, as was shown in Ref. 11 for a continuous Wigner distribution, and in the present study for discrete distributions, indirect methods for measuring them are possible. Positive definiteness of $P^{AA'B}_{\alpha\alpha'\beta}$ and of functions of this sort is not mandatory.

Just what would a negative joint probability mean? Dirac[1,2] saw in such a probability "a completely definite mathematical analog of a negative sum of money." We would add that, in a description of the results of the experiment we are considering here, a negative probability *reduces* the probability for events corresponding to it and *increases* the probability for opposite events. For example, we would have

$$\langle AB\rangle = P^{++}_{\alpha\beta} + P^{--}_{\alpha\beta} - P^{+-}_{\alpha\beta} - P^{-+}_{\alpha\beta}, \tag{25}$$

$$P^{++}_{\alpha\beta} = P^{+++}_{\alpha\alpha'\beta} + P^{+-+}_{\alpha\alpha'\beta}. \tag{26}$$

A negativity of $P^{+-+}_{\alpha\alpha'\beta}$ here means that the probability for the result $AB=+1$, a component of which is $P^{+-+}_{\alpha\alpha'\beta}$, decreases, while the probability for the opposite result $(AB=-1)$ increases. Like the Wigner distribution, multidimensional distributions of the $P^{AA'BB'}_{\alpha\alpha'\beta\beta'}$ type are convenient in calculations, and they put the results in a graphic form. Yet another advantage of such distributions is the possibility of an unambiguous resolution of Bell's paradox.[15]

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