Polarization shift of the levels of a muonic atom

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A polarization shift of the levels and a corresponding polarization potential has been found for muonic atoms with $Z \sim 10$–50, for which the Coulomb energy of the muon is on the order of or greater than the characteristic excitation energy of the nucleus.

Progress in the spectroscopy of muonic atoms[1] is stiffening the requirements on the accuracy of the corresponding atomic calculations, which must take into account progressively subtler physical effects (and thereby open up new sources of information about nuclear structure and fundamental interactions). One such effect is the polarization shift of atomic levels which results from virtual dipole excitations of the nucleus.[2]

Recent calculations of the polarization shift of muonic atoms[2–4] have been based on the assumption that the ratio (σ) of the Coulomb energy of the muon to the characteristic nuclear excitation energy is small:

$$\sigma \propto MR^2/ma_0^2 << 1. \tag{1}$$

Here $a_0 = \hbar^2/Ze^2m$ is the first Bohr radius, $-e$ and m are the charge and mass of the muon, Ze and R are the charge and radius of the nucleus, and M is the mass of the nucleon ($M \gg m$). Under condition (1) the motion of the virtual muon can be assumed free, and the polarization shift of the ground level (the most important level) can be written

$$\delta E = -\frac{4\sqrt{2m}\,e^2}{\pi\hbar a_0^3}\int_0^\infty \frac{d\omega}{\sqrt{\omega}}\,\mathrm{Im}\,\alpha(\omega)\,, \tag{2}$$

where $\alpha(\omega)$ is the dynamic polarizability of the nucleus.

Inequality (1) holds only for light nuclei, giving way to the opposite inequality as Z increases. It thus becomes necessary to generalize (2) to arbitrary values of σ; this generalization is the purpose of the present letter.

1. Under the condition $m^2e^2\langle \mathbf{d}^2\rangle/\hbar^4 \ll 1$, which means that the "charge-dipole" interaction $h = e\mathbf{d}\nabla(1/r)$ is weak (and under the obvious condition $R \ll a_0$), the polarization shift is given by

$$\delta E = \overline{< h(E+E' - H - H')^{-1}h >}\,. \tag{3}$$

Here the superior bar denotes the expectation value over the ground state of the muon, $\int d\mathbf{r}\psi^* \cdots \psi$; $H = p^2/2m - Ze^2/r$, E, and ψ are the Hamiltonian, energy, and wave function of the muon; H' and E' are the Hamiltonian and the energy of the motion within the nucleus; $\mathbf{d} = e\Sigma\vec{\rho}_i$ is the dipole momentum of the nucleus; $\vec{\rho}_i$ are the coordinates of the proton with respect to the center of mass of the nucleus; and the angle brackets denote an expectation value over the state of the nucleus.

The polarization shift can be expressed in terms of the Green's function of the muon, G:

$$\delta E = \frac{e^2\hbar}{\pi}\int_0^\infty d\omega \operatorname{Im}\alpha(\omega)\,\Lambda(\omega),$$
$$\Lambda(\omega) = \int d\mathbf{r}d\mathbf{r}'\psi^*(\mathbf{r})\nabla\, {}^1/_r G(\mathbf{r},\mathbf{r}'\mid E-\omega)\nabla\, {}^1/_{r'}\psi(\mathbf{r}'), \tag{4}$$

where

$$G=(E-H)^{-1}\delta(\mathbf{r}-\mathbf{r}'),\quad \operatorname{Im}\alpha(\omega) = \frac{\pi}{3}<\mathbf{d}\delta(H'-E'-\omega)\mathbf{d}>.$$

The representation[5]

$$G(\mathbf{r},\mathbf{r}'\mid E) = -\frac{m\kappa}{\hbar^2\pi}\int_1^\infty \frac{dt\,t^{1/\kappa a_0}}{(t-1)^2}\exp\left[-\kappa(r+r')\frac{t+1}{t-1}\right] I_0\left(\frac{2\kappa\sqrt{2t(rr'+\mathbf{r}\mathbf{r}')}}{t-1}\right),$$
$$\kappa = \sqrt{-2mE}/\hbar$$

leads to the expression $[\xi = (1+2ma_0^2\omega/\hbar)^{1/2}]$

$$\Lambda(\omega) = \frac{8m}{\hbar^2 a_0^2(\xi^2-1)}\left[\xi+1+2\Phi_{-1/\xi}\left[\left(\frac{\xi-1}{\xi+1}\right)^2\right]\right]. \tag{5}$$

The function $\Phi_\alpha(x)$, which can be expressed in terms of the hypergeometric function,

$$\Phi_\alpha(x) = \alpha^{-1} + x\Phi_{\alpha+1}(x) = \sum_{n=0}^\infty x^n/(n+\alpha) = \alpha^{-1}F(1,\alpha,\alpha+1,x)\quad (x<1),$$

has the asymptotic behavior

$$\Phi_\alpha(x) = \begin{cases} \alpha^{-1} + x/(\alpha+1) + x^2/(\alpha+2) + \ldots & (x \ll 1) \\ -\ln(1-x) + C + \psi(\alpha) + \cdots & (1-x \ll 1), \end{cases}$$

where ψ is the logarithmic derivative of the Γ function, and C is the Euler constant.

2. At this point, we switch to atomic units, $e = m = \hbar = 1$. Under condition (1), and in the lowest approximation in σ, we find (2) from (4) and (5); in the next approximation we find

$$\delta E = \frac{4Z^4}{\pi}\int_0^\infty \frac{d\omega}{\omega}\operatorname{Im}\alpha(\omega)\ln\left(\frac{\omega}{8Z^2}\right). \tag{6}$$

In particular, the polarization shift of the muonic atom of heavy hydrogen is[2]

$$\delta E = -\frac{Z\sigma^{3/2}}{M}\left[\frac{16}{105\pi} + \frac{\sigma^{1/2}}{64}\ln\sigma + \ldots\right], \tag{7}$$

where $\sigma = 2Z^2/\epsilon$, and ϵ is the binding energy of the deuteron.

In the opposite case, $\sigma \gg 1$, we find from the sum rules[6]

$$\delta E = -2Z^2 \langle \mathbf{d}^2 \rangle /3 + Z/2M - \langle \Sigma \mathbf{p}_i \mathbf{p}_j \rangle /2M^2 Z^2 + \langle \Sigma \nabla_i \nabla_j U \rangle /2M^2 Z^2 + \dots, \tag{8}$$

where $\mathbf{p}$ is the momentum of the proton, and U is the nuclear potential. The first term in (8), combined with the geometric level shift $2Z^2 \langle \Sigma \vec{\rho}_i^{\,2} \rangle /3$, caused by the distribution of the charge over the nucleus,[6] gives us

$$\delta E = Z^3 (Z-1) \chi /3, \tag{9}$$

where $\chi = \langle \Sigma (\vec{\rho}_i - \vec{\rho}_j)^2 \rangle / Z(Z-1)$ is the mean square distance between protons.

These expressions can be generalized to the case of an arbitrary bound state of the muon, $\psi(\mathbf{r})$, by introducing a factor $|\psi(0)|^2 \pi / Z^3$ in (2), (9), and the first terms in (7) and (8); by introducing a factor $-(\pi/2Z^4)(\partial |\psi|^2 / \partial r)|_0$ in (6) and the second term in (7); by introducing a factor $(\overline{1/r})/Z$ in the second and third terms in (8); and by introducing a factor $Z^2 \overline{r^2}/3$ in the fourth term in (8).

3. Using the Heisenberg equation $\ddot{\mathbf{r}} = -[H[H\mathbf{r}]] = Z\nabla {}^1/_r$ we can express the dynamic polarizability of a hydrogen-like atom in terms of the quantity $A(\omega)$ in (5) (Ref. 6):

$$A(\omega) = \frac{2}{3} \overline{\mathbf{r}(H-E)[(H-E)^2 - \omega^2 - i\delta]^{-1}\mathbf{r}}$$

$$= \frac{Z^2}{3\omega^4} [\Lambda(\omega) + \Lambda(-\omega - i\delta) - 3\omega^2/Z^2 - 4Z^2].$$

We find $A(0) = (9/2)Z^4$ and $A \to -\omega^{-2}$ in the limit $\omega \to \infty$, as we should; this behavior corresponds to the well-known high-frequency limit of the dielectric function of an atomic gas of density N: $\epsilon(\omega) = 1 + 4\pi N A(\omega) = 1 - 4\pi N/\omega^2$.

By the same approach we find the inverse relation

$$\Lambda(\omega) = \frac{3}{\pi Z^2} \int_0^\infty \frac{d\omega' \omega'^4}{\omega' + \omega} \operatorname{Im} A(\omega').$$

Comparison of this relation with (4) and with the known expression for the coefficient of the van der Waals forces between two complexes with polarizabilities α_1 and α_2,

$$\frac{6}{\pi^2} \int_0^\infty \frac{d\omega d\omega'}{\omega' + \omega} \operatorname{Im} \alpha_1(\omega) \operatorname{Im} \alpha_2(\omega')$$

we may interpret the polarization shift of the level as the result of a mutual polarization of two subsystems: the atomic shell and the nucleus.

4. Finally, we derive the polarization potential V which generates the polarization shift and which is given by (3) without the superior bar. If the excitation of the muon, $H - E$, is small, then (3) gives us the customary expression[1)] $V_0 = -e^2\alpha(0)/2r^4$. If, on the contrary, the nuclear excitation $H' - E'$, is small, then the identity $\nabla\psi = (H-E)^{-1} \nabla^{Ze^2}/_r \psi$, which holds for the discrete spectrum, gives us

$$V = \frac{<\mathbf{d}^2>}{3Zr^2} \frac{\partial}{\partial r} . \tag{10}$$

Using $\partial\psi/\partial r = -\psi/a_0$, we can convert (10) into the expression $V_1 = -\langle \mathbf{d}^2 \rangle / (3Za_0r^2)$, found in Ref. 2. Under the condition $\sigma \ll 1$, this expression holds at $R \ll r \ll \sqrt{M/m}R$, giving way to V_0 at larger values of r. If $\sigma \gg 1$, on the other hand, then $V = V_1$ at $R \ll r \ll a_0$; $V = V_0$ at $r \gg MR^2/ma_0$; and general expression (10) must be used in the intermediate region. This expression also applies to the relativistic case, in which the nuclear excitation energy is above mc^2 (in particular, for an electronic atom[4]). In this case, potential (10) holds at $R \ll r \ll MR^2c/\hbar$, giving way to V_0 at larger values of r.

[1)]Ordinary units are being used in this section.

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