Integration of the nonlinear dynamics of a uniaxial ferromagnet by the method of the inverse scattering problem

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The method of the inverse scattering problem is used to investigate the nonlinear dynamics of ferromagnets with easy-axis and easy-plane anisotropies.

1. The one-dimensional dynamics of a ferromagnet with a preferred-axis anisotropy is described by a Landau–Lifshitz (L–L) equation of the form (the waves propagate along the preferred axis)

$$\vec{\mu}_t = \vec{\mu} \times \vec{\mu}_{zz} + 4\beta^2(\vec{\mu} \times \mathbf{n})(\vec{\mu}\,\mathbf{n}), \tag{1}$$

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where $\vec{\mu} = \mathbf{M}/|\mathbf{M}|$; $\mathbf{M}$ is the density vector of the magnetic moment of the medium; $t = g|\mathbf{M}|\tilde{t}$, $\tilde{t}$ is the time, and g is the gyromagnetic ratio; $z = \alpha^{-1/2}\tilde{z}$, $\tilde{z}$ is the spatial variable and α is the inhomogeneous exchange interaction constant; β^2 is a constant determined by the energy of anisotropic interaction in the crystal; and, $\mathbf{n}$ is a unit vector along the preferred axis (z axis).

The construction of exact solutions of Eq. (1) has been discussed in many papers (see, for example, Ref. 1), but the method of the inverse scattering problem (MISP), which in our view gives the greatest amount of information on the solutions of the nonlinear equation of evolution being investigated, has not yet been applied to the L–L equation, in spite of the fact that an infinite number of Lax representations has been found for this equation.[2] In this paper we construct a closed MISP structure for each, physically different sign of the constant β^2 in the L–L equation (1).

2. Ferromagnet with an easy-axis anisotropy ($\beta^2 > 0$)

As shown in Ref. 3, if $\beta^2 > 0$, Eq. (1) is guage-equivalent (Zakharov's context) to a nonlinear Schrödinger equation (the case of attraction), which has been integrated by Zakharov and Shabat.[4] As the Lax pair we shall choose the pair[2] having the form

$$\varphi_z^\alpha = A_1^{\alpha\beta}\varphi^\beta \qquad A_1 = i\begin{pmatrix} \lambda\mu\,; & (\lambda - i\beta)\mu^- \\ (\lambda + i\beta)\mu^+\,; & -\lambda\mu \end{pmatrix}, \tag{2a}$$

$$\varphi_t^\alpha = B_1^{\alpha\beta}\varphi^\beta \qquad B_1 = i\begin{pmatrix} 2(\lambda^2+\beta^2)\mu + \lambda\tau\,; & (\lambda - i\beta)(2\lambda\mu^- + \tau^-) \\ (\lambda + i\beta)(2\lambda\mu^+ + \tau^+)\,; & -2(\lambda^2+\beta^2)\mu - \lambda\tau \end{pmatrix}, \tag{2b}$$

where $\mu = \mu^3$; $\mu^\pm = \mu^1 \pm i\mu^2$; $2i\tau = \mu_x^+\mu^- - \mu^+\mu_x^-$; $\pm i\tau^\pm = \mu^\pm\mu_x - \mu\mu_x^\pm$; and λ is a free parameter. In the dynamics of an easy-axis ferromagnet it would be logical to examine the state with $\mu = 1$; $\mu^\pm = 0$; $|z| \to \infty$. The unperturbed linear equation, generated by the matrix $A_1(|z| = \infty) = i\lambda\sigma_3$, has a fundamental matrix of solutions $\Phi = \exp(i\lambda\sigma_3 z)$ and the spectral problem (2a) has an involution, $\varphi = \sigma_2\varphi^*\sigma_2$ (σ_i are the Pauli matrices). Let $\varphi^\pm$ be the fundamental matrices of Jost's solutions (2a) ($\varphi^\pm \to \Phi$ as $z \to \pm\infty$, respectively). From the involution and $Sp\,A_1 = 0$ we then find

$$\varphi^- = \varphi^+ T_1\,; \qquad T_1 = \begin{pmatrix} a_1^*\,; & b_1 \\ -b_1^*\,; & a_1 \end{pmatrix} \qquad \det T_1 = 1. \tag{3}$$

The first column $\varphi^+ - \varphi_1^+$ and the second column $\varphi^- - \varphi_1^-$ of the Jost's matrix are analytic in the upper λ half-plane and have the asymptotic forms

$$\varphi_1^+ e_1^- \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad \varphi_2^- e_1^+ \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \quad e_1^\pm = \exp(\pm i\lambda z)\,; \quad |z| \to \infty\,; \quad \mathrm{Im}\,\lambda > 0,$$

which are also analytic in the upper half-plane, and the roots of $a_1(\lambda^k) = 0$ correspond to the discrete spectrum. Equation (2b) determines the evolution of the scattering data,

$$a_1(\lambda\,;t) = a_1(\lambda\,;0)\,; \qquad b_1(\lambda\,;t) = b_1(\lambda\,;0)\exp[4i(\lambda^2+\beta^2)t\,].$$

The matrix of Jost's solutions has a triangular representation

$$\varphi^+(z;\lambda) = \Phi(z;\lambda) + \int_z^\infty K_1(z;s;\lambda)\Phi(s;\lambda)\,ds\,;\; K_1 = \begin{pmatrix}\lambda K_1^1\,; & (i\beta-\lambda)K_1^{2*}\\ (\lambda+i\beta)K_1^2; & \lambda K_1^{1*}\end{pmatrix},$$

where the functions K_1^1, K_1^2, which do not depend on $\tilde{\lambda}$, are the solution of the problem

$$K^1_{1,zs}(z;s) = -\mu K^1_{1,ss}(z;s) - \mu^- K^2_{1,ss}(z;s) + \beta^2\mu^- K_1^2(z;s), \tag{4a}$$

$$K^2_{1,z}(z;s) = \mu K^2_{1,s}(z;s) - \mu^+ K_1^1(z;s)$$

with conditions on the diagonals

$$i\mu^- K_1^2(z;z) = [1 + iK_1^1(z;z)](1-\mu),$$

$$iK_1^2(z;z)(1+\mu) = [1 + iK_1^1(z;z)]\mu^+\,, \tag{4b}$$

$$K^1_{1,z}(z;s=z) = -\mu K^1_{1,s}(z;s=z) - \mu^- K^2_{1,s}(z;s=z).$$

Integrating (3) along the real axis λ with the factor $(2\pi)^{-1}\exp(i\lambda y)$, we obtain Marchenko's integro-differential equations

$$iF_1(z+y) + K^{2*}_{1,y}(z;y) + \beta K_1^{2*}(z;y) - \int_z^\infty K_1^1(z;s)F_{1,y}(y+s)ds = 0,$$

$$K^{1*}_{1,y}(z;y) + \int_z^\infty K_1^2(z;s)\{F_{1,y}(y+s) - \beta F_1(y+s)\}ds = 0, \tag{5}$$

$$F_1(z) = \int_{-\infty}^{\infty}\frac{d\lambda}{2\pi}\exp(i\lambda z)b_1/a_1 - i\sum_k \exp(i\lambda^k z)b_1(\lambda^k)/a_{1,\lambda}(\lambda^k).$$

The solutions of (5) are determined up to arbitrary function f_1, $\tilde{K}_1^2 = f_1K_1^2; \tilde{K}_1^1 = f_1K_1^1 + i(1-f_1)$. However, from the conditions on the diagonal (4b) it follows that μ and $\mu^\pm$ do not depend on the real f_1 function. Substituting μ and $\mu^\pm$ found from (4b) into the third condition on the diagonal (4b), we obtain an equation for the f_1 function, and the MISP is closed. We obtain the soliton solutions by examining nonreflective potentials ($b_1(\lambda) = 0$) in the associated scattering problem (2a) with complex eigenvalues λ^k. Incorporation of an external magnetic field oriented along the anisotropy axis does not fundamentally change the method.

3. Ferromagnet with an easy-plane anisotropy

We shall investigate the states with $\mu(|z| = \infty) = 0$, $\mu^\pm(z = \infty) = 1$ and $\mu^\pm(z = -\infty) = \exp(\pm ic)$, where c is a constant, in an easy-plane ferromagnet. Lax's pair in this case has the form[2]

$$\psi_z^\alpha = A_2^{\alpha\beta}\psi^\beta\,; \qquad A_2 = i\begin{pmatrix}\lambda\mu\,; & (\lambda+\beta)\mu^-\\ (\lambda-\beta)\mu^+; & -\lambda\mu\end{pmatrix}, \tag{6a}$$

$$\psi_t^{\alpha} = B_2^{\alpha;\beta}\,\psi^{\beta}\,; \qquad B_2 = i\begin{pmatrix} 2(\lambda^2-\beta^2)\mu+\lambda\tau\,; & (\lambda+\beta)(2\lambda\mu^-+\tau^-) \\ (\lambda-\beta)(2\lambda\mu^++\tau^+); & -2(\lambda^2-\beta^2)\mu-\lambda\tau \end{pmatrix}. \tag{6b}$$

The unperturbed equations $\psi_z = A_2^{\pm}\psi$; $A_2^{+} = A_2(z=\infty)$; $A_2^{-} = A_2(z=-\infty)$ have the following fundamental solution matrices

$$\Psi^+ = \begin{pmatrix} e_2^+; & \zeta e_2^-/(\beta-\lambda) \\ \zeta e_2^+/(\beta+\lambda)\,; & e_2^- \end{pmatrix}; \qquad \Psi^- = \begin{pmatrix} \exp(-ic)\,e_2^+; & \zeta e_2^-/(\beta-\lambda) \\ \zeta e_2^+/(\beta+\lambda); & \exp(ic)\,e_2^- \end{pmatrix}.$$

The spectral problem in (6a) has the involutions $\psi = (\beta\sigma_1 + i\lambda\sigma_2)\psi^*(\beta\sigma_1 + i\lambda\sigma_2)^{-1}$, and the fundamental Jost matrices $\psi^{\pm} \to \Psi^{\pm}$ and $z \to \pm\infty$ are related by

$$\psi^- = \psi^+ T_2; \qquad T_2 = \begin{pmatrix} a_2^*\,; & (\beta+\lambda)b_2 \\ (\beta-\lambda)b_2^*; & a_2 \end{pmatrix}; \qquad \det T_2 = 1. \tag{7}$$

The first column ψ^+ and the second column ψ^- of the Jost matrix are analytic in the upper sheet of the two-sheet Riemann surface $\zeta = (\lambda^2 - \beta^2)^{1/2}$; $a_2(\lambda)$ is also analytic in the upper sheet of this surface; and, the roots of $a_2(\lambda^k) = 0$ determine the discrete spectrum of problem (6a). The evolution of the scattering data follows from (6b): $a_2(\lambda; t) = a_2(\lambda; 0)$; $b_2(\lambda; t) = b_2(\lambda; 0)\exp(4i\lambda\zeta t)$. The matrix of Jost's solutions has a triangular representation of the form

$$\psi^+(z;\lambda) = \Psi^+(z;\lambda) + \int_z^{\infty} K_2(z;s;\lambda)\,\Psi^+(s;\lambda)\,ds; \quad K_2 = \begin{pmatrix} \lambda K_2^1\,; & (\beta+\lambda)K_2^2 \\ (\beta-\lambda)K_2^{2*}\,; & \lambda K_2^{1*} \end{pmatrix}, \tag{8}$$

where the functions K_2^1 and K_2^2, which are independent of λ, are the solution of the problem,

$$K^2_{2,zs}(z;s) = -\mu K^1_{2,ss}(z;s) + \mu^+ K^{2*}_{2,ss}(z;s) + \beta^2\mu K_2^1 \tag{9a}$$

$$K^1_{2,z}(z;s) = -\mu K^2_{2,s}(z;s) - \mu^- K^{1*}_{2,s}(z;s),$$

$$-i + K_2^2(z;z) = -i\mu^- + \mu K_2^1(z;z) - \mu^- K_2^{2*}(z;z);$$

$$-K_2^1(z;z) = i\mu - \mu^- K_2^{1*}(z;z) - \mu K_2^2(z;z),$$

$$K^2_{2,z}(z;s=z) = -\mu K^1_{2,s}(z;s=z) + \mu^- K^{2*}_{2,s}(z;s=z). \tag{9b}$$

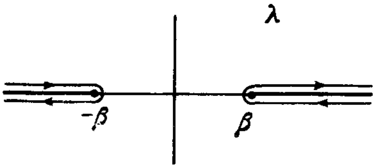

FIG. 1.

Integrating (7) over the contour (illustrated in Fig. 1) with integrating factors $\exp(i\zeta y)/4\pi(\lambda+\beta)$; $\lambda\exp(i\zeta y)/4\pi(\lambda+\beta)$, we find the Marchenko equations

$$F_2^1(z+y)-K_2^1(z;y)+\int_z^\infty\{K_2^1(z;s)F_2^2(y+s)-iK_2^2(z;s)F^1_{2,y}(y+s)\}ds=0,$$

$$F_2^2(z+y)-iK^2_{2,y}(z;y)+\int_z^\infty\{K_2^1(z;s)[\beta^2F_2^1(y+s)-F^1_{2,yy}(y+s)]-$$

$$-iK_2^2(z;s)F^2_{2,y}(y+s)\}ds=0,$$

where

$$F_2^1(z)=\int_{-\infty}^{\infty}\frac{\zeta d\zeta}{4\pi\lambda}\exp(i\zeta z)[b_2(\lambda)/a_2(\lambda)-b_2(-\lambda)/a_2(-\lambda)$$

$$-\frac{i}{2}\Sigma\exp(i\zeta^k z)b_2(\lambda^k)/a_{2,\lambda}(\lambda^k),$$

$$F_2^2(z)=\int_{-\infty}^{\infty}\frac{\zeta d\zeta}{4\pi}\exp(i\zeta z)[b_2(\lambda)/a_2(\lambda)+b_2(-\lambda)/a_2(-\lambda)]$$

$$-\frac{i}{2}\Sigma\exp(i\zeta^k z)\lambda^k b_2(\lambda^k)/a_{2,\lambda}(\lambda^k).$$

The solutions of (10) are determined to within an arbitrary function f_2, $\widetilde{K}_2^1=f_2K_2^1$; $\widetilde{K}_2^2=f_2K_2^2+i(1-f_2)$. From the conditions on the diagonal (9b) it follows that μ and $\mu^\pm$ do not depend on f_2 and, analogously to the preceding case, the MISR is closed. The solitons in spectral problem (6a) correspond to nonreflective potentials with a real discrete spectrum, and bions correspond to nonreflective potentials with a pair of complex conjugate eigenvalues. The construction developed here can be extended to the case of an external magnetic field oriented along the anisotropy axis.

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[1)]In this paper, a letter subscript indicates the corresponding partial derivative.

[2)]In the case of a ferromagnet with an easy-plane anisotropy, an external-field-dependent unperturbed operator A_2 ($|z|=\infty$) appears and the state of the magnet as $|z|\to\infty$ depends strongly on the field.

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