Nonperturbative three-point vertex function and magnetic mass of the gluon

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A nonperturbative three-gluon vertex function which satisfies the Slavnov-Taylor identities and which has a series of exact limits is found in the $A_4 = 0$ axial gauge. A self-consistent equation is constructed for the "magnetic mass" of the gluon, and this equation is solved. In this approximation, the magnetic mass of the gluon is identically equal to zero.

Nonperturbative calculations of Green's functions have been pursued actively in recent years and are continually being refined. They are particularly important for the theory of non-Abelian gauge fields, in which the infrared difficulties are not eliminated by conventional calculations. Nonperturbative calculation methods, which are ordinarily constructed in an axial gauge through the use of the exact Schwinger-Dyson representation and the Slavnov-Taylor identities, effectively make it possible to sum the perturbation-theory series and to bring out the nonanalytic behavior of the quantities of interest. The axial gauge is distinguished by the simple form of the Slavnov-Taylor identities and apparently represents the only type of gauge which is suitable for the development of a nonperturbative calculation method. A difficulty of these calculation methods which has yet to be overcome is the ambiguity of their closure, which stems from the uncertainty regarding the choice of a structure for the nonperturbative vertex function. The reason for this uncertainty is that the Slavnov-Taylor identities do not determine the transverse part of this vertex function. Furthermore, the gauge dependence of all the calculations complicates a comparison of the results obtained by different approaches. As an example of this ambiguity we might cite the nonperturbative calculations of the "magnetic mass" of the gluon; this mass should apparently be on the order of[1,2] g^2T, but the values which have been found so far differ widely, depending on the method used for the nonperturbative calculations. Some studies[3] put

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this mass on the order of $g^{3/2}T$, but there are others[4] which show that the magnetic mass of the gluon is zero to an accuracy of $g^{3/2}T$.

The vertex functions which have been constructed satisfy the Slavnov-Taylor identities exactly and reproduce the single-loop perturbation-theory calculations. The axial gauge is fixed, and the ordinary Yang-Mills theory for SU(N) groups is studied. The Slavnov-Taylor identities are at their simplest for this class of gauges[5]:

$$r_\mu \Gamma_{\mu ij}(r,p,q)^{abc} = igf^{abc}\{ D_{ij}^{-1}(p) - D_{ij}^{-1}(q) \} \tag{1}$$

They allow closure of the self-consistent calculations with non-perturbative diagrams of a single topological form. The calculations are carried out in a Euclidean metric ($k^2 = \mathbf{k}^2 + k_4^2$), and the subclass of gauges with $A_4 = 0$ is selected. The gluon propagator is determined by two scalar functions,[3]

$$D_{ij}^{-1}(k) = \left(1 + \frac{G(k)}{k^2}\right)(k^2\delta_{ij} - k_i k_j) + \frac{F(k) - G(k)}{k^2}\,\frac{k_4^2}{\mathbf{k}^2}\,k_i k_j \tag{2}$$

and has a simple tensor structure. The seed vertex function is written in the conventional notation as

$$\Gamma^{(0)}_{\mu\nu\gamma}(r,p,q)^{abc} = -\,igf^{abc}[\,\delta_{\mu\nu}(r-p)_\gamma + \text{antisymm.}\,] \tag{3}$$

and satisfies (1) when we have $F = G = 0$ in (2) and when the total angular momentum is conserved, $r + p + q = 0$.

Both vertex functions, Γ_{4ji} and Γ_{eji}, are defined in terms of the structure functions of the gluon propagator, and neither has any kinematic singularities. The vertex Γ_{4ji},

$$\Gamma_{4ji}(q,r,p)^{abc} = -\,igf^{abc}\left\{\delta_{ij}(r_4 - p_4)\left(1 - \frac{\dfrac{F(r)}{r^2} - \dfrac{F(p)}{p^2}}{r^2 - p^2}\,r_4 p_4\right) + \delta_{ij}\left(r_4\frac{F(r)}{r^2} - p_4\frac{F(p)}{p^2}\right) - \frac{\left(\dfrac{G(r)}{\mathbf{r}^2} - \dfrac{F(r)}{\mathbf{r}^2}\dfrac{r_4^2}{r^2}\right) - \left(\dfrac{G(p)}{\mathbf{p}^2} - \dfrac{F(p)}{\mathbf{p}^2}\dfrac{p_4^2}{p^2}\right)}{r^2 - p^2}\right.$$

$$\times(\mathbf{pr}\delta_{ij} - p_j r_i)(r_4 - p_4)$$

$$\left. + \frac{\dfrac{F(q)}{q^2} - \dfrac{F(r)}{r^2}}{q^2 - r^2}\,q_j r_4 (q-r)_i + \frac{\dfrac{F(p)}{p^2} - \dfrac{F(q)}{q^2}}{p^2 - q^2}\,q_i p_4 (p-q)_j\right\}, \tag{4}$$

is used for nonperturbative calculations of Π_{44} and ultimately determines the function $F(k)$. The vertex Γ_{eji} is the primary structural unit of the nonperturbative calculation method. The structure of Γ_{eji} is consistent with identities (1),

$$\Gamma_{eji}(q,r,p)^{abc} = -igf^{abc}\Bigg\{\delta_{ij}\left(1-\frac{\dfrac{F(r)}{r^2}-\dfrac{F(p)}{p^2}}{r^2-p^2}\,r_4p_4\right)(r_e-p_e)$$

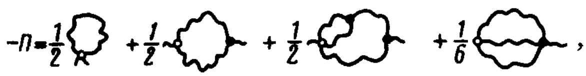
$$-\frac{\left(\dfrac{G(r)}{\mathbf{r}^2}-\dfrac{F(r)}{\mathbf{r}^2}\dfrac{r_4^2}{r^2}\right)-\left(\dfrac{G(p)}{\mathbf{p}^2}-\dfrac{F(p)}{\mathbf{p}^2}\dfrac{p_4^2}{p^2}\right)}{r^2-p^2}(\mathbf{p}\mathbf{r}\,\delta_{ij}-p_jr_i)(r_e-p_e)$$

$$+\delta_{ij}\left[r_e\left(\frac{G(r)}{\mathbf{r}^2}-\frac{F(r)}{\mathbf{r}^2}\frac{r_4^2}{r^2}\right)-\left(\frac{G(p)}{\mathbf{p}^2}-\frac{F(p)}{\mathbf{p}^2}\frac{p_4^2}{p^2}\right)p_e\right]$$

$$+\ \text{antisymm.}\Bigg\} \qquad (5)$$

and is similar to the structure of Γ_{ij4}. The antisymmetric part in (5) is constructed in a manner identical to that in (3). In this nonperturbative calculation approach, the vertex functions Γ_{444} and Γ_{4i4} simply maintain the Slavnov-Taylor identities and do not participate directly in the self-consistent equations which determine the structure functions of the gluon propagator.

Nonperturbative calculations of the functions $F(k)$ and $G(k)$ use a diagram representation for the polarization operator,



(6)

and nonperturbative expressions for the Green's functions and the vertex functions determining (6). All the methods which have been developed for self-consistent calculations use only the first two diagrams; this simplification is justified in an axial gauge, where there are no fictitious particles. It is usually assumed that incorporating the other diagrams would not cause any qualitative changes in the calculated results. When expressions (4) and (5) are taken into account, the calculation method is closed, and the nonperturbative expression for the gluon propagator is found as the solution of the two coupled equations for the functions $F(k)$ and $G(k)$.

The magnetic mass of the gluon is the most interesting result to emerge from such calculations. The calculation method is closed by means of the vertex function Γ_{eij}, for which an expression is written in accordance with (5) but in the infrared limit ($q_4=0, |\mathbf{q}|\to 0$):

$$\Gamma_{eji}(0,-p,p)^{abc} = \left(1+\frac{G(p)}{\mathbf{p}^2}\right)\Gamma_{eji}(0,-p,p)^{abc}$$

$$+ igf^{abc}\left\{\left(\frac{F(p)}{\mathbf{p}^2}\frac{p_4^2}{p^2}\right)[(\delta_{ei}p_j+\delta_{ej}p_i)-2\hat{p}_i\hat{p}_j p_e]+(\mathbf{p}^2\delta_{ij}\right.$$

$$\left.-p_ip_j)\hat{p}_e\left(\frac{\partial}{\partial|\mathbf{p}|}\frac{G(p)}{\mathbf{p}^2}\right)+\left(\frac{\partial}{\partial|\mathbf{p}|}\frac{F(p)}{\mathbf{p}^2}\right)\frac{p_4^2}{\mathbf{p}^2}p_ip_j\hat{p}_e\right\}. \quad (7)$$

This expression for the function Γ_{eji}, where $\hat{p}_e = p_e/|\mathbf{p}|$, is the same as its exact asymptotic expression,[6]

$$\Gamma_{eji}(0,-p,p)^{abc} = igf^{abc}\frac{\partial D_{ij}^{-1}(p)}{\partial p_e} \quad (8)$$

which is found through an independent differentiation of the "inverse" gluon propagator (2). Other limits for expressions (4) and (5) have also been checked. Their equivalence to the known asymptotic expressions have been clearly established, but for a strictly defined order in which the limits are taken. In a calculation of the diagrams one uses a standard form for the nonperturbative Green's function in the $A_4=0$ gauge[3]:

$$D_{ij}(k) = \frac{1}{k^2+G(k)}\left(\delta_{ij}-\frac{k_ik_j}{\mathbf{k}^2}\right)+\frac{1}{k^2+F(k)}\frac{k^2}{k_4^2}\frac{k_ik_j}{\mathbf{k}^2} \quad (9)$$

This expression is found by inverting function (2). A self-consistent equation for the magnetic mass of the gluon follows from the definition of the function $G(k)$:

$$m_{mag}^2 \equiv G(0,0) = \frac{1}{2}\sum_i \Pi_{ii}(0,0), \quad (10)$$

where all the limiting expressions are understood in the infrared limit ($q_4=0, |\mathbf{q}|\to 0$). After the polarization operator is calculated, Eq. (10) takes the form of a nonlinear integral equation,

$$m_{mag}^2 = g^2NT\sum_{q_4}\int\frac{d^3q}{(2\pi)^3}\left[\frac{q^2}{q_4^2(q^2+F(q))}\right.$$

$$+\frac{2}{q^2+G(q)}-\left(1+\frac{G(q)}{q^2}\right)\left(\frac{2\mathbf{q}^2}{[q^2+G(q)]^2}+\frac{q^2}{q_4^2}\frac{1}{q^2+G(q)}\frac{\mathbf{q}^2}{q^2+F(q)}\right)$$

$$\left.+\frac{F(q)-G(q)}{[q^2+G(q)][q^2+F(q)]}-q^2|\mathbf{q}|\frac{\partial}{\partial|\mathbf{q}|}\left(\frac{G(q)}{q^2}\right)\frac{1}{[q^2+G(q)]^2}\right], \quad (11)$$

in which the first two terms correspond to the "tadpole" in series (6), while the other terms correspond to the second diagram. A series of identity transformations of Eqs. (11) eliminates the function $F(k)$:

$$m^2_{mag} = g^2 NT \sum_{q_4} \int \frac{d^3 q}{(2\pi)^3} \left(3 + |\mathbf{q}| \frac{\partial}{\partial |\mathbf{q}|}\right) \frac{1}{q^2 + G(q)} \tag{12}$$

reducing the problem to simply a self-consistent calculation of the function $G(k)$. We recall that Eq. (11) is constructed from the first two terms in series (6) but is otherwise exact.

Equation (12) cannot be solved analytically. However, under the additional assumption

$$\lim_{|\mathbf{q}|} |\mathbf{q}|^3 T \sum_{q_4} \frac{1}{\mathbf{q}^2 + q_4^2 + G(\mathbf{q}, q_4)} \Bigg|_0^\infty = 0, \tag{13}$$

which allows us to integrate by parts in Eq. (12), summed over frequency, we can immediately find a solution:

$$m^2_{mag} \equiv 0, \tag{14}$$

which is unique. This solution becomes analytically exact if we assume at the outset that the function $G(q_4,\mathbf{q})$ can be replaced by $G(0,q)$ in the denominator in (12).

The result $m^2_{\rm mag} \equiv 0$, is exact for the nonperturbative-calculation method which we have used here. In this sense, it summarizes all the other work which has been carried out to calculate $m^2_{\rm mag}$ in the $A_4 = 0$ gauge. The complete agreement of result (14) with the conclusions of Refs. 4 is further testimony to its validity, while the contradiction with other studies[3] can be explained on the basis that these other studies have worked from an expression for the three-gluon vertex function which is not consistent with the Slavnov-Taylor identities. On the whole, however, the situation is still unclear. We cannot definitely say that the magnetic mass of the gluon is a gauge-invariant quantity, and it is not completely clear whether the approximation used here is correct. Further study of this problem would be definitely worthwhile at this point, although it is clear even at this point that future nonperturbative calculation methods which use the $A_4 = 0$ gauge will be far more complicated. In particular, the calculations will require an approximation of the four-gluon vertex and of the nonperturbative expression for the three-gluon vertex for all momenta, as found in (4) and (5).

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