Layered structure of superfluid ^{4}He with supercritical motion

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It is shown that when superfluid ^{4}He flows along a capillary with a velocity exceeding Landau's critical roton velocity, a one-dimensional periodic structure, which is at rest relative to the walls, appears in the helium and the spectrum of excitations is deformed so that the criterion of superfluidity is not violated.

Landau's criterion plays an important role in the modern theory of superfluidity. According to this criterion, superfluid motion is possible if

$$\tilde{\epsilon}(\mathbf{p}) \equiv \epsilon(p) + \mathbf{pv} > 0 \tag{1}$$

along the entire curve of the spectrum $\epsilon(p)$ of elementary excitations.[1] (We are examining motion in a laboratory system of coordinates, in which the walls of the vessel and the normal part of the liquid are at rest.) For the actual spectrum of ^{4}He (curve 1 in Fig. 1) this leads to the condition

$$v < v_c , \tag{2}$$

where v_c is equal to the tangent of the slope angle of the tangent to the roton part of the spectrum ($v_c \approx 60$ m/s). The question of what happens to the liquid when this velocity is exceeded, as far as we know, remains unclear. We shall show that for small excesses above the critical velocity v_c, a one-dimensional periodic structure, which is at rest relative to the walls, appears in the helium with a wave vector oriented opposite the flow and equal to $p_c/\hbar$, where $p_c \sim p_0$ is the momentum at the tangent point. In this case the spectrum of excitations is deformed so that criterion (1) is nowhere violated.

The above formulation of the problem requires some important stipulations. As is well known, thus far it has not been possible to attain the critical velocity (2). The reason lies in the creation of excitations of another type—vortical rings. The maximum velocities—about 8–10 m/s—were obtained with helium flowing through a small opening with a diameter of 5–20 μm.[2,3] It appears that an improvement of the experimental technique will permit achieving higher velocities, exceeding (2). The author hopes that this paper will stimulate further investigations in this direction.

The quantity $\tilde{\epsilon}$ defined by Eq. (1) is the energy of excitation in the liquid moving with velocity $\mathbf{v}$. Inequalities (1) or (2) ensure that $\tilde{\epsilon}$ is positive. If $\tilde{\epsilon}$ becomes negative, then an unlimited creation of boson excitations—rotons—becomes energetically favorable. Here the boson distribution function $n(\tilde{\epsilon})$ becomes negative, indicating the impossibility of thermodynamic equilibrium of the ideal gas of rotons.

The following arguments are easier to understand if a graph of $\tilde{\epsilon}$ as a function of p_x is constructed [the x axis is oriented along $(-\mathbf{v})$]. Curve 2 in Fig. 1 corresponds to

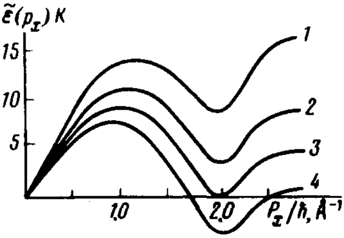

FIG. 1.

values $v < v_c$, curve 3 corresponds to values $v = v_c$, and curve 4 corresponds to values $v > v_c$. The similarity of the behavior of $\tilde{\epsilon}$ to the behavior of the frequency of the "soft mode" near the second-order phase-transition point is evident. We believe that this analogy has a deep significance: at $v = v_c$ the helium undergoes a second-order phase transition from a spatially homogeneous state to a layered state. At $v = v_c$ the distribution function becomes infinite at the point $\mathbf{p} = \mathbf{p}_c$ (the vector $\mathbf{p}_c$ is oriented along the x axis, x, $|\mathbf{p}_c| = p_c$). The density of the superfluid part, however, remains finite. A simple calculation gives

$$\frac{\rho_n(T, v_c)}{\rho} \approx 0,08 \frac{p_0^4}{\Delta^2} \frac{\mu^{1/3}}{\hbar^3} \frac{T^{3/2}}{\rho} \approx (T, \mathrm{K})^{3/2} \tag{3}$$

(Δ, p_0, μ are the parameters of the roton spectrum). [In calculating (3) it is, of course, necessary to use a Bose, and not a Boltzman, distribution function for the rotons.]

We shall write out the Hamiltonian of the roton gas, corresponding to the energy (1), in the form $\int \hat{\psi}[\epsilon(p) + \mathbf{pv}]\hat{\psi} d^3x$, where $\hat{\psi} - \psi$ is a roton operator, and $\mathbf{p}$ is an operator acting on ψ. Since for $v > v_c$ equilibrium in the gas of noninteracting rotons is impossible, the interaction between them must be taken into account. It is sufficient to take the interaction operator in the form $(g/2)\int \hat{\psi}^+ \hat{\psi}^+ \hat{\psi}\hat{\psi} d^3x$. In what follows, the sign of the constant g is important. Experimental data on neutron scattering in helium apparently indicate that $g > 0$, $g \approx 2 \times 10^{-38}$ erg · cm^3.[4] The final form of the energy operator of the system is

$$\hat{H} = \int \{ \hat{\psi}^+ [\epsilon(p) + \mathbf{pv}] \hat{\psi} + \frac{g}{2} \hat{\psi}^+ \hat{\psi}^+ \hat{\psi}\hat{\psi} \} d^3x \, . \tag{4}$$

Terms that are cubic in ψ should not be written out in (4). They describe three-roton processes, in which at least one roton has a momentum which is not close to $\mathbf{p}_c$, and only the interaction of "critical" rotons is important.

When the critical velocity v_c is reached, creation of rotons with momentum $\mathbf{p}_c$ and their accumulation in this state begins.[1)] However, ψ, an operator of the system in which there is a large number of bosons in a single state, can be viewed simply as a classical function of the coordinates.[5] We shall seek it in the form of a plane wave

$$\psi_0 = \eta \exp(i\mathbf{p}_c \mathbf{r}/\hbar), \tag{5}$$

determining the amplitude η from the condition that energy (4) is minimized. Taking into account the fact that $\epsilon(p_c) = p_c v_c$, we write (4) in a form analogous to the free

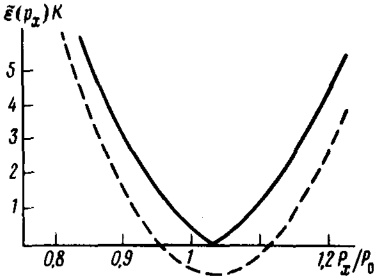

FIG. 2.

energy of the usual theory of second-order phase transitions,

$$\int [p_c (v_c - v)|\eta|^2 + \frac{g}{2}|\eta|^4] d^3x. \tag{6}$$

Minimizing (6), we find $|\eta|^2 = (v - v_c)p_c/g$ for $v > v_c$. The plane wave (5) corresponds to a spatially uniform distribution of rotons. It is remarkable, however, that it leads to a spatial modulation of the density of the liquid, since the density operator $\hat{n}$ contains a term which is linear in the operator $\hat{\psi}$:

$$\hat{n} = n_0 + \sqrt{n_0}\,(A\hat{\psi} + A^*\hat{\psi}^+). \tag{7}$$

This term is responsible for the creation of a roton with scattering of a neutron in helium and the coefficient A determines the probability of this process. The corresponding contribution to the dynamic form factor of the liquid is $|A|^2\delta[\epsilon - \epsilon(p)]$. A reasonable estimate of A can be obtained assuming that this term is the main contribution to the Placzek sum rule. We then find $|A|^2 = p_c^2/2m\epsilon(p_c) \approx 2.5$.

Replacing $\hat{\psi}$ in (7) by the classical wave (5), we find that the density of helium is modulated according to the law

$$\frac{n - n_0}{n_0} = \left[\frac{|A|^2(v - v_c)p_c}{n_0 g}\right]^{1/2} \sin(p_c x) \approx 2.6\left[\frac{v - v_c}{v_c}\right]^{1/2} \sin(p_c x). \tag{8}$$

The appearance of such a one-dimensional density wave is the main prediction of the theory expounded here. It can be observed, in principle, in the experiments on scattering of x-rays in moving helium.

The roton spectrum changes for $v > v_c$. To find it we must substitute $\hat{\psi} = \psi_0 + \hat{\psi}'$ for $\hat{\psi}$ in (4), retaining terms that are quadratic in $\hat{\psi}'$. The problem then actually coincides with Bogolyubov's problem for a nonideal Bose gas and the spectrum has the form[5]

$$\epsilon(\mathbf{p}) = \sqrt{2p_c(v - v_c)\epsilon_0(\mathbf{p}) + [\epsilon_0(\mathbf{p})]^2}, \quad \epsilon_0(\mathbf{p}) = \Delta + \frac{(p - p_0)^2}{2\mu} - p_x v_c. \tag{9}$$

We can see that the minimum value of $\tilde{\epsilon}$ is zero, so that Landau's criteria is nowhere violated. The spectrum is anisotropic and the velocity of the excitations near the zero of $\tilde{\epsilon}$ is finite. The dependence of $\tilde{\epsilon}$ on p_x for $(v - v_c)/v_c = 0.1$ is shown in Fig. 2.

The above picture of the phase transition with respect to the flow velocity is also valid at finite temperatures. The similarity noted above between (6) and the free energy in the theory of second-order phase transitions leads to the conclusion that the dependence of the normal density on the velocity corresponds to the dependence of the entropy on the temperature near the transition point. In the fluctuation region this gives

$$\rho_n(v) = \rho_n(v_c) + B(v_c - v)^{1-\alpha}, \tag{10}$$

where α is the critical index of the heat capacity.

The quantitative analysis presented above is based on the assumption that $g > 0$. If, however, $g < 0$, then the qualitative picture of the phenomenon remains unchanged. As before, there will be some critical velocity above which a density wave appears in the helium. The transition in this case, however, is a first-order phase transition.

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[1)]Following Ref. 6, the accumulation of rotons in a state with momentum $\mathbf{p}_c$ can be viewed as their Bose condensation.

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