Analysis of the behavior of the pion form factor at small Q^2 by the method of QCD sum rules

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A QCD sum rule has been derived for the electromagnetic form factor of the pion in the region of a small spacelike momentum transfer, $Q^2 \lesssim m_\rho^2$. The value found for the electromagnetic radius of the pion, $\langle r_\pi^2 \rangle 1/2 = 0.66 \pm 0.03$ fm, agrees well with experimental data.

The quantum-chromodynamics sum rules[1] are actually the only calculation method available in QCD for problems in which nonperturbative effects play a governing role. Analysis of the various two-point functions $\pi(p^2)$ (correlation functions) by this method has yielded values for the masses and lepton widths of mesons and the corresponding characteristics of baryons.[2] The method was extended to three-point functions in Refs. 3 and 4. In particular, the behavior of the electromagnetic form factor of the pion was calculated for intermediate values of the momentum transfer, $m_\rho^2 \lesssim Q^2 \lesssim 3$–$4$ GeV^2, through an analysis of the amplitude $T(Q^2, p_1^2, p_2^2)$:

$$T(Q^2, p_1^2, p_2^2) = \frac{n_\mu n_\alpha n_\beta}{2(nP)^3} i^2 \int d^4x\, d^4y < 0 \,|\, T\{j_\alpha^+\left(-\frac{y}{2}\right) j_\beta\left(\frac{y}{2}\right) J_\mu(x)\} \,|\, 0 > \cdot \exp\left(i\,\frac{p_1 + p_2}{2}\, y - iqx\right) \tag{1}$$

where $j_\alpha = \bar{u}\gamma_5\gamma_\alpha d$, $J_\mu = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d$ and n is a lightlike vector orthogonal to $q = p_2 - P_1 : n^2 = 0\ (nq) = 0, (np_1 + (np_2 \equiv (nP)]$. This analysis was carried out in a symmetric kinematics, $|p_1^2| \sim |p_2^2 \sim Q^2 \gtrsim m_\rho^2$ (where $Q^2 = -q^2$). The approach of Refs. 3 and 4 cannot be used at small values of Q^2 because the amplitude $T(Q^2, p_1^2, p_2^2)$ contains contributions which are strongly dependent on the dynamics over distances $\sim 1/Q$. At $Q^2 \lesssim m_\rho^2 = 0.6$ GeV^2 we cannot use asymptotic freedom (i.e., perturbation theory) to evaluate such contributions.

To refine the estimates (i.e., to incorporate nonperturbative effects) we need to carry out an additional factorization of the contributions from small and large distances. Problems of this type have been taken up previously in calculations by the method of QCD sum rules of the magnetic moments of nucleons[5,6] and of the constant[7] g_A. Our own approach to the factorization of the contributions is based on the technique of Ref. 8 for analyzing the asymptotic behavior of Feynman diagrams. Using this technique, we easily find that in the kinematics $|p_1^2| \sim |p_2^2| \gg Q^2$ the contributions to $T(Q^2, p_1^2, P_2^2)$, which are proportional to powers of $1/p_1^2$ and $1/p_2^2$, arise not only from the region in which all the intervals $[y^2, (x+y/2)^2$, and $(x-y/2)^2]$ are small but also from the region where y^2 is small but $(x \pm y/2)^2$ is not. Taking this second possibility into account gives us some additional contributions with the structure

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$C(p^2)[\pi(q^2) - \pi^{\text{pert}}(q^2)]$, where $\pi(q^2)$ is the "exact" correlation function of a local operator with a current J, and π^{pert} is its perturbative analog (cf. Ref. 6). The incorporation of a contribution of this type evidently corresponds to replacing the perturbative estimate of the contribution $C(p^2)\pi(q^2)$, which depends strongly on the dynamics at distances $\sim 1/Q$, by an expression incorporating nonperturbative effects. Technically, this replacement is achieved by using the dispersion relation

$$\pi(q^2) - \pi^{\text{pert}}(q^2) = \frac{1}{\pi}\int_0^\infty \frac{ds}{s-q^2}(\rho(s) - \rho^{\text{pert}}(s)), \tag{2}$$

where $\rho(s)$ is the "exact" spectral density and $\rho^{\text{pert}}(s)$ the perturbative spectral density. The reason why there are no subtractions in (2) is that we have $\pi(q^2) - \pi^{\text{pert}}(q^2) \to 0$ in the limit $|q^2| \to \infty$ by virtue of asymptotic freedom (see Appendix B in Ref. 9). Specific calculations of $\rho(s)$ have used the model $\rho(s) = f\delta(s - m_\rho^2) + \rho^{\text{pert}}(s)\theta(s > s_0)$ whose parameters (f and s_0) have been extracted in the standard way from the QCD sum rules[1] for the corresponding correlation function $\pi(q^2)$. For correlation functions with $\rho^{\text{pert}}(s) = 0$ this approximation reduces to vector dominance (cf. Ref. 6). For the $\langle\bar{\psi}\psi\rangle^2$ corrections, however, which make a numerically large contribution, we need to refine this approximation by parametrizing the contribution of higher-lying states by means of an effective resonance, (i.e., by taking[10] $\rho(s) = f_1\delta(s - m_\rho^2) + f_2\delta(s - m_{\rho'}^2)$). The sum rule found by us in this manner for $\hat{B}(p^2, M^2)T(Q^2, p_1^2, p_2^2)$ [$\hat{B}(p^2, M^2)$ is a Borel transformation[1]] is

$$\frac{f_\pi^2 F_\pi(Q^2)}{M^4} + \frac{c_\pi(Q^2)}{M^2} + e^{-m_{A_1}^2/M^2}\left\{\frac{f_{A_1}^2 F_{A_1}(Q^2)}{M^4} + \frac{c_{A_1}(Q^2)}{M^2}\right\} + \dots$$

$$= \frac{3}{2\pi^2 M^2}\int_0^1 dx \int_0^1 d\xi\, x(1-x)\exp\left(-\frac{x}{1-x}\xi(1-\xi)\frac{Q^2}{M^2}\right)$$

$$+ \frac{1}{40\pi^2 M^6}\left[Q^4 \ln\left(\frac{Q^2}{Q^2+s_0}\right) + Q^2 s_0 - \frac{s_0^2}{2}\frac{Q^2}{Q^2+m_\rho^2}\right]$$

$$+ \frac{\alpha_s \langle G^a_{\mu\nu} G^a_{\mu\nu}\rangle}{24\pi M^6}\left(1 + \left[\frac{m_\rho^2}{Q^2+m_\rho^2}\right]\right)$$

$$+ \frac{16}{81}\frac{\pi\alpha_s\langle\bar{\psi}\psi\rangle^2}{M^8}\left(5 + 6\left[\frac{1.6\, m_\rho^2}{Q^2+m_\rho^2} - \frac{0.6\, m_{\rho'}^2}{Q^2+m_{\rho'}^2}\right]\right), \tag{3}$$

where $s_0 = 1.5$ GeV^2, $m_{\rho'}^2 = 2.0$ GeV^2, and $m_{A_1}^2 = 1.6$ GeV^2.

We would like to point out two properties of sum rule (3).

a) At $Q^2 = 0$, the right side of (3) (multiplied by M^2) agrees with the expansion for the correlation function of axial currents, $\pi \sim n^\alpha n^\beta \langle j_\alpha^+ j_\beta\rangle/(nP)^2$, so that we have

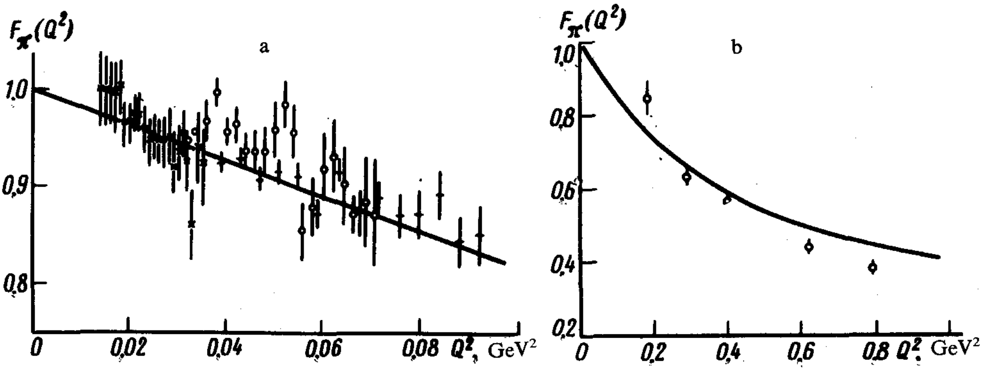

FIG. 1. Comparison of the experimental data on $F_\pi(Q^2)$ with the results of an analysis of sum rule (3). a—The region $Q^2<0.1$ GeV^2, with data from Ref. 11; b—the region $Q^2 \leqslant 1$ GeV^2, with data from Ref. 12.

$F_\pi(0)=1$, $F_{A_1}(0)=1$, and $C_i(0)=0$. This fact is a consequence of the Ward identity for T (cf. Ref. 7). Subtracting from (3) the sum rule for $Q^2=0$, we find a sum rule which can be used to find $[F_\pi(Q^2)-1]/Q^2$. Following the standard procedure for analyzing a sum rule of this type, 5–7 we find the value $\langle r_\pi^2\rangle 1/2 = 0.66 \pm 0.33$ fm for the electromagnetic radius of the pion, in good agreement with the experimental value $\langle r_\pi^2\rangle^{1/2}_{\rm expt} = 0.636 \pm 0.037$ fm (Fig. 1a).

b) The additional terms which we mentioned earlier [enclosed in square brackets in (3)] must "die out" with increasing Q^2. It is easy to see that this requirement is met in (3). We note, however, that an extrapolation of the $1/Q^2$ decay law of these terms beyond the formal range of applicability of sum rule (3), $Q^2 \lesssim m_\rho^2$, would be incorrect, since there are no contributions $\mathscr{O}(1/Q^2)$ (or of higher powers of $1/Q^2$) in the $\langle GG\rangle$ and $\langle\bar\psi\psi\rangle^2$ corrections in the operator expansion for $T(Q^2, p_1^2, p_2^2)$ in the kinematics $Q^2 \sim |p^2|$. In other words, at $Q^2 \gtrsim m_\rho^2$ the additional contributions should actually fall off more rapidly than any power of $1/Q^2$. Since these terms in (3) do not fall off rapidly enough, the value found for $F_\pi(Q^2=0.6\ \mathrm{GeV}^2)$ turns out to be 10% higher than the experimental value (Fig. 1b). On the other hand, if we set the additional contributions equal to zero (this approach corresponds to switching to a sum rule for the kinematics $Q^2 \sim |p^2| \gtrsim m_\rho^2$), we find a value $F_\pi^{\rm teor}(Q^2=0.6\ \mathrm{GeV}^2)$ which is 10% smaller than the experimental value. This result means that the sum rules for the two kinematics ($Q^2 \ll m_\rho^2$ and $Q^2 \gtrsim m_\rho^2$) at the point $Q^2 = m_\rho^2$ agree with each other and with the experimental curve to the necessary accuracy of 10–20%.

In summary, the method of QCD sum rules has been used to generate a good description of the data on the electromagnetic form factor of the pion over essentially the entire region of space-like momentum transfer which has been studied experimentally, $0 \leqslant Q^2 \leqslant 3-4$ GeV^2.

In the course of our calculations we learned of a result derived by Chetyrkin *et al.*,[13] who attempted to calculate $\langle r_\pi^2\rangle$ by an analogous method but without considering the $\langle GG\rangle$ and $\langle\bar\psi\psi\rangle^2$ contributions, whose calculations were the most laborious part of our own study. Incorporating these contributions increases the value found in Ref. 13 for $\langle r_\pi^2 \rangle$ by a factor of 1.5, but the point we wish to make is not this numerical discrepancy but the circumstance that it is incorrect in principle to discard contributions of this sort when using the method of QCD sum rules, since these contributions accumulate information on the dynamics of the system in question.

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[1]M. A. Shifman, A. I. Vaĭnshteĭn, and V. I. Zakharov, Nucl. Phys. **B147**, 385, 448 (1979).
[2]B. L. Ioffe, Nucl. Phys. **B188**, 317 (1981).
[3]V. A. Nesterenko and A. V. Radyushkin, Pis'ma Zh. Eksp. Teor. Fiz. **35**, 395 (1982) [JETP Lett. **35**, 488 (1982)]; V. A. Nesterenko and A. V. Radyushkin, Phys. Lett. **115**, 410 (1982).
[4]B. L. Ioffe and A. V. Smilga, Phys. Lett. **114B**, 353 (1982).
[5]B. L. Ioffe and A. V. Smilga, Pis'ma Zh. Eksp. Teor. Fiz. **37**, 250 (1983) [JETP Lett. **37** 298 (1983)].
[6]I. I. Balitsky and A. V. Yung, Phys. Lett. **129B**, 388 (1983).
[7]V. M. Belyaev and Ya I. Kogan, Pis'ma Zh. Eksp. Teor, Fiz. **37**, 611 (1983) [JETP Lett. **37**, 730 (1983)].
[8]A. V. Efremov and A. V. Radyushkin, Teor. Mat. Fiz. **44**, 16, 157, 327 (1980); **42**, 147 (1980); A. V. Radyushkin, Fiz. Elem. Chastits At.. Yadra. **14**, 58 (1983) [Sov. J. Part. Nucl. **14**, 23 (1983)].
[9]A. I. Vaĭnshteĭn *et al.*, Fiz. Elem. Chastits At. Yadra. **13**, 542 (1982) [Sov. J. Part. Nucl. **13**, 224 (1982)]; V. A. Novikov *et al.*, Nucl. Phys. **B191**, 301 (1981).
[10]V. M. Belyaev and Ya. Ĭ. Kogan, ITEP-12, Moscow, 1984.
[11]A. S. Vodop'yanov and É. N. Tsyganov, Fiz. Elem. Chastits At. Yadra. **15**, 5 (1984) [Sov. J. Part. Nucl. **15** (to be published)].
[12]C. Bebek *et al.*, Phys. Rev. **D17**, 1693 (1978).
[13]K. G. Chetyrkin, S. G. Gorishny, A. B. Krasulin, S. A. Larin, and V. A. Mateveev, Report *P*-340, INR, Moscow, 1984.

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