Dirac monopole as a Lagrangian system in a stratification space

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A simple realization of a global Lagrangian description of the relative motion of a monopole and a charge is given. This realization is based on a transfer of the dynamics to the space of a stratification which is equivalent to a Hopf stratification.

Wu and Yang[1] and Greub and Petry[2] have proposed a description of the quantum chromodynamics of a charged particle in the field of a monopole in terms of a stratification theory which excludes the Dirac string of singularities. The stratifications of Refs. 1 and 2 are labeled by the integer n the Dirac condition $2e\mu = n$. They are constructed[2,3] from a Hopf stratification (more precisely, from a stratification which is homotopically equivalent to the Hopf stratification). Balachandran *et al.*[4] have shown that the transformation to a stratification space is possible and useful even in classical mechanics. Our purpose in the present letter is to obtain a realization of a Lagrangian system on the basis of the stratification[5] $\dot{C}^2/U$, which will allow the clearest and most compact representation of both the classical dynamics of this problem and the quantization. The group quantization technique of Ref. 6 is important in the realization $R^1\times SU(2)/U(1)$ adopted in Ref. 4. In the present letter we manage to avoid that technique.

We begin with a few words on the meaning of transforming to a stratification space. This transformation means expanding configuration space by adding a gauge degree of freedom. Only after this has been done can the overall monopole-charge system have a Lagrangian description, albeit with a degenerate Lagrangian. The system itself is not a Lagrangian system. The function

$$\frac{m}{2}\dot{\mathbf{x}}^2 + e\mathbf{A}\dot{\mathbf{x}} \tag{1}$$

which is ordinarily used as the Lagrangian depends on the gauge, is singular, and is invariant under rotations. The corresponding action is a multivalued functional of the trajectory, and the variational principle must be understood in a local sense.[4,7,8] In contrast, the Lagrangian in a stratification space is defined globally and is invariant; the corresponding action is single-valued. The degeneracy of the Lagrangian in the Hamiltonian description corresponds to a primary-coupling condition, which leads to the relation $2e\mu = n$ and to a stratification series upon quantization.[1,2]

Let us consider the space C^2 of pairs of complex numbers $(z_1 = z_1 e^{i\varphi_1}, z_2 = r_2 e^{i\varphi_2})$. We use the notation $\bar{z}\zeta = \bar{z}_1\zeta_1 + \bar{z}_2\zeta_2$, and we define the projection onto ordinary space (R^3) by the expression $x_i = \bar{z}\sigma_i z$, where the σ_i are the Pauli matrices. In spherical coordinates we have $r = r_1^2 + r_2^2$, $\theta = 2\text{arc ctg}(r_1/r_2)$, and $\varphi = \varphi_2 - \varphi_1$. Pairs of the form $(z_1 e^{i\alpha}, z_2 e^{i\alpha})$, $e^{i\alpha}\in U(1)$ are projected to a common point x. In other words, if we remove the point which is the position of the monopole, $x=0$, then we find that $\dot{R}^3 = R^3 - \{0\}$ serves as a set of orbits onto which the $\dot{C}^2 = C^2 - \{0\}$ action of the gauge group $U(1)$ stratifies. We consider the Lagrangian

$$L = \frac{m}{2}[4(\bar{z}z)(\dot{\bar{z}}\dot{z}) + (\dot{\bar{z}}z - \bar{z}\dot{z})^2] + ie\mu\frac{\dot{\bar{z}}z - \bar{z}\dot{z}}{\bar{z}z} . \quad (2)$$

It is easy to see that if we specify some local cross section [for example, if we set $r_1 = r^{1/2}\cos(\theta/2)$, $r_2 = r^{1/2}\sin(\theta/2)$, $\varphi_1 = 0$, $\varphi_2 = \varphi$], then (2) transforms into (1) with the A of the corresponding gauge [in the present case, $A_r = A_\theta = 0$, $A_\varphi = (\mu/r)\tan(\theta/2)$]. The sum of angular velocities appears linearly in (2) with a coefficient $e\mu$, and the corresponding Lagrange-Euler equation takes the form of the identity $0=0$; i.e., the motion along the gauge coordinate $\chi = \varphi_1 + \varphi_2$ is completely undefined. Lagrangian (2) thus describes the dynamics of orbits, rather than of an individual point z. The other equations when projected onto R^3 give us the ordinary equations of motion of a charge in the field of a monopole. We adopt the notation $p = \partial L/\partial\dot{\bar{z}}, \bar{p} = \partial L/\partial\dot{z}$. Using a Legendre transformation, we find the coupling surface

$$\bar{z}p - z\bar{p} = 2ie\mu , \quad (3)$$

on which the Hamiltonian

$$H = \frac{1}{2m\bar{z}z}\left(\bar{p}p - \frac{e^2\mu^2}{\bar{z}z}\right) \quad (4)$$

is defined. The limitation of the simplectic form onto surface (3) is degenerate along the gauge orbits. After a factorization in terms of these orbits we find the Hamiltonian of the system in $\dot{R}^3$, but for this Hamiltonian there are no global canonical coordinates.[4,8]

Returning to configuration space, we consider the space $(\hat{\Omega})$ of closed trajectories in $\dot{C}^2$ whose initial and final points are z_0. The action S defined by Lagrangian (2) is a single-valued functional on these trajectories. In addition to the trajectories $z(t)$, $t \in [0,1]$, we consider its projection $x(t)$ and also the mapping of the segment $[0,1]$ into a circle, which associates with the point t the gauge coordinate $\chi(t)$. If $z(t)$ and $z'(t)$ have a single projection, and the corresponding mappings into a circle cover the circle an identical number of times, then we have $S(z) = S(z')$. The factor space of the space $\hat{\Omega}$ is singly connected from the standpoint of this equivalence and serves as a covering space for the space (Ω) of trajectories $x(t)$, $x(0) = x(1) = x_0$. Its role is analogous to that played by a Riemannian surface in the theory of analytic functions: A transformation to it converts an initially multivalued action functional into a single-valued functional.[8]

Lagrangian (2) is explicitly invariant under the action of the stratification of the SU(2) group in the space:

$$z \to uz, \qquad u \in SU(2). \quad (5)$$

This action is a stratification automorphism and induces an action $SO(3) = SU(2)/Z_2$ in $\dot{R}^3$. The transformation $u = \exp(-ian_i\sigma_i)$ corresponds to a rotation through an angle $2a$ around the axis **n**. The invariance of Lagrangian (2) under transformations (5) thus explicitly expresses the symmetry of the problem under rotations. This symmetry is formally lost in the standard formulation, (1). Corresponding to symmetry (5) of Lagrangian (2) is the conserved quantity

$$\frac{1}{2i}(\bar{p}\,\sigma_i z - \bar{z}\sigma_i p). \tag{6}$$

With this choice of coefficients, this quantity is exactly equal to the angular momentum $J_i = m\varepsilon_{ijk}x_j\dot{x}_k - e\mu x_i/r$.

We turn now to the quantization. We replace p by $-i\partial/\partial\bar{z}$, and we replace $\bar{p}$ by $-i\partial/\partial z$, in Eqs. (3), (4), and (6). Coupling condition (3) takes the form $\partial/\partial\chi = ie\mu$ and gives us a specific functional dependence of the wave function on the gauge coordinate:

$$\Psi(z) = e^{ie\mu\chi}\,\psi(r_1, r_2, \varphi). \tag{7}$$

Since the wave function must not change when φ_1 and φ_2 are simultaneously increased by 2π, we find a Dirac condition on the charges: $2e\mu = n$. Expression (7) can now be written

$$\Psi(ze^{i\alpha}) = e^{in\alpha}\,\Psi(z), \quad e^{i\alpha} \in U(1). \tag{8}$$

If $|n| > 1$, the wave function does not change when the angles are increased by $2\pi/n$; i.e., the wave function is actually defined in the factor space $\dot{C}^2/Z_n$, which is homotopically equivalent[3] to a lens L_n. Under the natural definition of the action of the gauge group, $[z]e^{i\alpha} = [ze^{-i\alpha/n}]$, this action converts into a stratification for which (8) serves as an equivariance condition.[2] A series of stratifications is reproduced in this manner.[1,2] The action (5) dictates the following representation of the rotation group in the space of wave functions:

$$\Psi(z) \to \Psi(u^{-1}z).$$

A representation of this sort was studied in Ref. 9. We see that the operator (6) is its generator. For even values of n, this representation is single-valued, since the matrix $u = -I$ does not displace points of the lens with an even index, and SO(3) actually operates. For odd values of n representation (9) is double-valued, and the eigenvalues of operator (6) are half-integers.

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